You too can experience rapid rotation—if your stomach can take the high angular velocity and centripetal acceleration. If not, try the slower ferris wheel. Rotating carnival rides have rotational KE as well as angular momentum.

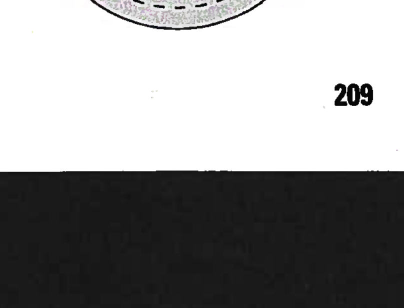
CHAPTER

ROTATIONAL MOTION 8

We have, until now, been concerned mainly with translational motion. In this chapter, we will deal with rotational motion. We will mainly be concerned with rigid bodies. By a **rigid body** we mean a body with a definite shape that doesn't change, so that the particles composing it stay in fixed positions relative to one another. Of course, any real body is capable of vibrating or deforming when a force is exerted on it. But these effects are often very small, so the concept of an ideal rigid body is very useful as a good approximation.

The motion of a rigid body (as mentioned in Chapter 7) can be analyzed as the translational motion of its center of mass, plus rotational motion about its center of mass. We have already discussed translational motion in detail, so now we focus our attention on purely rotational motion. By *purely rotational motion*, we mean that all points in the body move in circles, such as the point P in the rotating wheel of Fig. 8–1, and that the centers of these circles all lie on a line called the **axis of rotation**, which in Fig. 8–1 is perpendicular to the page and passes through point O.

FIGURE 8–1 Looking down on a wheel that is rotating counterclockwise about an axis through the wheel's center at O (perpendicular to the page). The dashed circular line is the path of point P.



8–1 Angular Quantities

To describe rotational motion, we make use of angular quantities, such as angular velocity and angular acceleration. These are defined in analogy to the corresponding quantities in linear motion.

Every point in a body rotating about a fixed axis moves in a circle (shown dashed in Fig. 8–1 for point P) whose center is on the axis and

Each particle moves in a circle, and each sweeps out the same angle

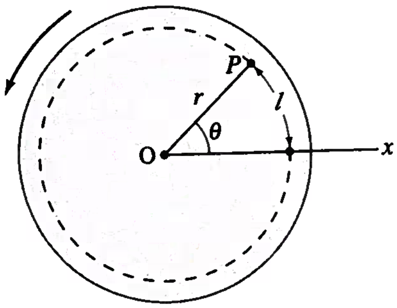


FIGURE 8–1 (Repeated.)

whose radius is r, the distance of that point from the axis of rotation. A straight line drawn from the axis to any point sweeps out the same angle θ in the same time.

To indicate the angular position of the body, or how far it has rotated, we specify the angle θ of some particular line in the body with respect to some reference line, such as the x axis (see Fig. 8–1). A point in the body (such as P in Fig. 8–1) moves through an angle θ when it travels the distance l measured along the circumference of its circular path. Angles are commonly measured in degrees, but the mathematics of circular motion is much simpler if we use the *radian* for angular measure. One **radian** (rad) is defined as the angle subtended by an arc whose length is equal to the radius. For example, in Fig. 8–1, point P is a distance r from the axis of rotation, and it has moved a distance l along the arc of a circle. The arc length l is said to "subtend" the angle θ. If $l = r$, then θ is exactly equal to 1 rad. In general, any angle θ is given by

θ in radians

$$\theta = \frac{l}{r}, \qquad \textbf{(8–1)}$$

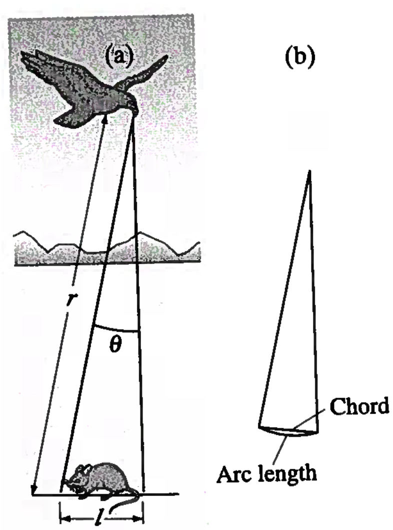
where r is the radius of the circle, and l is the arc length subtended by the angle θ which is specified in radians. Radians can be related to degrees in the following way. In a complete circle there are 360°, which of course must correspond to an arc length equal to the circumference of the circle, $l = 2\pi r$. Thus $\theta = l/r = 2\pi r/r = 2\pi$ rad in a complete circle, so

$$360° = 2\pi \text{ rad}.$$

1 rad ≈ 57.3°

One radian is therefore $360°/2\pi \approx 360°/6.28 \approx 57.3°$.

FIGURE 8–2 (a) Example 8–1. (b) For small angles, arc length and the chord length (straight line) are nearly equal.



EXAMPLE 8–1 **Birds of prey—in radians.** A particular bird's eye can just distinguish objects that subtend an angle no smaller than about 3×10^{-4} rad. (*a*) How many degrees is this? (*b*) How small an object can the bird just distinguish when flying at a height of 100 m (Fig. 8–2a)?

SOLUTION (*a*) One radian is $360°/2\pi$, so 3×10^{-4} rad is

$$(3 \times 10^{-4}\text{ rad})\left(\frac{360°}{2\pi\text{ rad}}\right) = 0.017°.$$

(*b*) From Eq. 8–1, $l = r\theta$. For small angles, the arc length and the chord length are approximately† the same (Fig. 8–2b). Since $r = 100$ m and $\theta = 3 \times 10^{-4}$ rad, we find

$$l = (100\text{ m})(3 \times 10^{-4}\text{ rad}) = 3 \times 10^{-2}\text{ m} = 3\text{ cm}.$$

Had the angle been given in degrees, we would first have had to change it to radians to make this calculation.

Note in this Example that we used the fact that the radian is dimensionless (has no units) since it is the ratio of two lengths.

†Even for an angle as large as 15°, the error in making this estimate is only 1 percent, but for larger angles the error increases rapidly.

When an object, such as the bicycle wheel in Fig. 8–3, rotates from some initial position, specified by θ_0, to some final position, θ, its angular displacement is $\Delta\theta = \theta - \theta_0$. The *angular velocity* is defined in analogy with ordinary linear velocity. Instead of linear displacement, we use the angular displacement. Thus the **average angular velocity** (denoted by ω, the Greek lowercase letter omega) is defined as

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}, \tag{8–2a}$$

where $\Delta\theta$ is the angle through which the body has rotated in the time Δt. We define the **instantaneous angular velocity** as the very small angle, $\Delta\theta$, through which the body turns in the very short time interval Δt:

Angular velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}. \tag{8–2b}$$

Angular velocity is generally specified in radians per second (rad/s). Note that *all points in a rigid body rotate with the same angular velocity*, since every position in the body moves through the same angle in the same time interval.

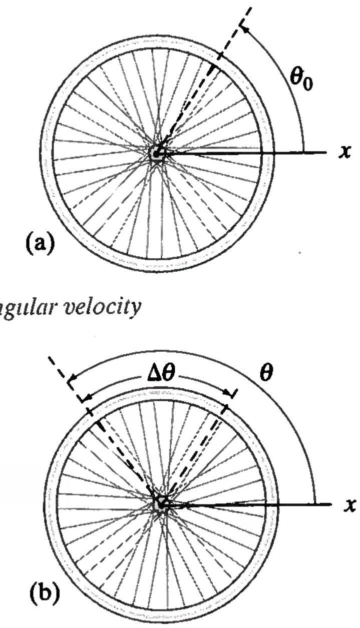


FIGURE 8–3 A wheel rotates from (a) initial position θ_0, to (b) final position θ. The angular displacement is $\Delta\theta = \theta - \theta_0$.

Angular acceleration, in analogy to ordinary linear acceleration, is defined as the change in angular velocity divided by the time required to make this change. The **average angular acceleration** (denoted by α, the Greek lowercase letter alpha) is defined as

$$\overline{\alpha} = \frac{\omega - \omega_0}{\Delta t} = \frac{\Delta\omega}{\Delta t}, \tag{8–3a}$$

where ω_0 is the angular velocity initially, and ω is the angular velocity after a time Δt. **Instantaneous angular acceleration** is defined in the usual way:

Angular acceleration

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}. \tag{8–3b}$$

Since ω is the same for all points of a rotating body, Eq. 8–3 tells us that α also will be the same for all points. Thus, ω and α are properties of the rotating body as a whole. With ω measured in radians per second and t in seconds, α will be expressed as radians per second squared (rad/s^2).

Each particle or point of a rotating rigid body has, at any moment, a linear velocity v and a linear acceleration a. We can relate these linear quantities, v and a, of each particle, to the angular quantities, ω and α, of the rotating body as a whole. Consider a particle located a distance r from the axis of rotation, as in Fig. 8–4. If the body rotates with angular velocity ω, any particle will have a linear velocity whose direction is tangent to its circular path. The magnitude of its linear velocity, v, is $v = \Delta l/\Delta t$. From Eq. 8–1, a change in rotation angle $\Delta\theta$ is related to the linear distance traveled by $\Delta l = r\,\Delta\theta$. Hence

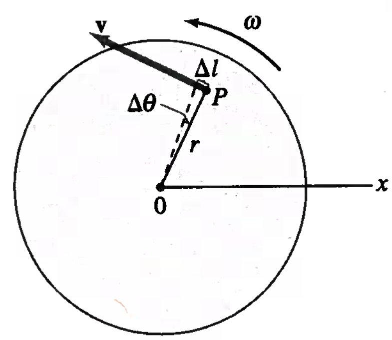


FIGURE 8–4 A particle P on a rotating wheel has a linear velocity **v** at any moment.

$$v = \frac{\Delta l}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

or

Linear and angular velocity related

$$v = r\omega. \tag{8–4}$$

Thus, although ω is the same for every point in the rotating body at any

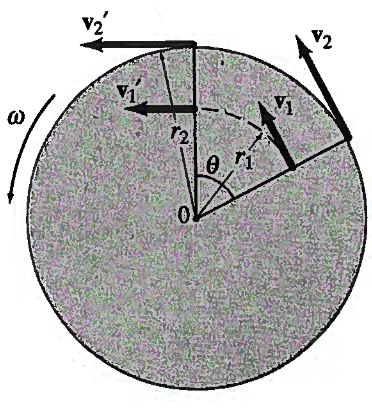


FIGURE 8–5 A wheel rotating uniformly counterclockwise. Two points on the wheel, at distances r_1 and r_2 from the center, have different linear velocities because they travel different distances in the same time interval. Since $r_2 > r_1$, then $v_2 > v_1$ ($v = r\omega$). But the two points have the same angular velocity ω because they travel through the same angle θ in the same time interval.

instant, the linear velocity v is greater for points farther from the axis (Fig. 8–5). Note that Eq. 8–4 is valid both instantaneously and on the average.

CONCEPTUAL EXAMPLE 8–2 **Is the lion faster than the horse?** A rotating carousel has one child sitting on a horse near the outer edge and another child seated on a lion halfway out from the center. (*a*) Which child has the greater linear speed? (*b*) Which child has the greater angular speed?

RESPONSE (*a*) The *linear* speed is the distance traveled divided by the time interval. In one rotation the child on the outer edge travels a longer distance than the child near the center, but the time interval is the same for both. Thus the child at the outer edge has the greater linear speed.
(*b*) The *angular* speed is the angle of rotation divided by the time interval. In one rotation both children rotate through the same angle ($360° = 2\pi$ radians). Thus the two children have the same angular speed.

We can use Eq. 8–4 to show that the angular acceleration α is related to the tangential linear acceleration a_{tan} of a particle in the rotating body by

$$a_{tan} = \frac{\Delta v}{\Delta t} = r\frac{\Delta \omega}{\Delta t}$$

or

Tangential acceleration

$$a_{tan} = r\alpha. \tag{8–5}$$

In this equation, r is the radius of the circle in which the particle is moving, and the subscript tan in a_{tan} stands for "tangential" since the acceleration considered here is along the circle (that is, tangent to it).

The total linear acceleration of a particle is the vector sum of two components:

$$\mathbf{a} = \mathbf{a}_{tan} + \mathbf{a}_R,$$

where the radial† component, $\mathbf{a}_R$, is the radial or "centripetal" acceleration and points toward the center of the particle's circular path (Fig. 8–6). We saw in Chapter 5 (Eq. 5–1) that $a_R = v^2/r$, and we rewrite this in terms of ω using Eq. 8–4:

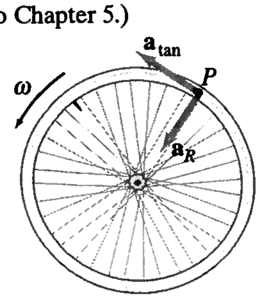
Centripetal (or radial) acceleration

$$a_R = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r. \tag{8–6}$$

Thus the centripetal acceleration is greater the farther you are from the axis of rotation: the children farthest out on a merry-go-round feel the greatest acceleration. Equations 8–4, 8–5, and 8–6 relate the angular quantities describing the rotation of a body to the linear quantities for each particle of the body.

We can relate the angular velocity ω to the frequency of rotation, f, where by **frequency** we mean the number of complete revolutions (rev)

FIGURE 8–6 On a rotating wheel whose rotation speed is increasing, a point P has both tangential and radial (centripetal) components of acceleration. (See also Chapter 5.)



†"Radial" means along the radius—that is, toward or away from the center or axis.

per second. One revolution (of a wheel, say) corresponds to an angle of 2π radians, and thus $1\ \text{rev/s} = 2\pi\ \text{rad/s}$. Hence, in general, the frequency f is related to the angular velocity ω by

Frequency

$$f = \frac{\omega}{2\pi}$$

or

$$\omega = 2\pi f. \qquad \textbf{(8–7)}$$

The unit for frequency, revolutions per second (rev/s), is given the special name, the hertz (Hz). That is

$$1\ \text{Hz} = 1\ \text{rev/s}.$$

Note that "revolution" is not really a unit, so we can also write $1\ \text{Hz} = 1\ \text{s}^{-1}$.

The time required for one complete revolution is called the **period**, T, and it is related to the frequency by

$$T = \frac{1}{f}. \qquad \textbf{(8–8)}$$

Period

For example, if a particle rotates at a frequency of three revolutions per second, then each revolution takes $\frac{1}{3}$ s.

EXAMPLE 8–3 Speed and acceleration on a merry-go-round. (*a*) What is the linear speed of a child seated 1.2 m from the center of a steadily rotating merry-go-round (Fig. 8–7) that makes one complete revolution in 4.0 s? (*b*) What is her acceleration?

SOLUTION (*a*) First, we find the angular velocity in radians per second: the period is given as 4.0 s, so

$$f = \frac{1}{T} = \frac{1\ \text{rev}}{4.0\ \text{s}} = 0.25\ \text{rev/s} = 0.25\ \text{Hz}.$$

Then

$$\omega = 2\pi f = \left(2\pi\frac{\text{rad}}{\text{rev}}\right)\left(0.25\frac{\text{rev}}{\text{s}}\right) = 1.6\ \text{rad/s}.$$

The radius r is 1.2 m, so the speed v is

$$v = r\omega = (1.2\ \text{m})(1.6\ \text{rad/s}) = 1.9\ \text{m/s}.$$

(*b*) Since $\omega = 1.6\ \text{rad/s} = \text{constant}$, then $\alpha = 0$ and the tangential component of the linear acceleration (Eq. 8–5) is

$$a_{\tan} = r\alpha = 0.$$

From Eq. 8–6, the radial component is

$$a_R = \omega^2 r = (1.6\ \text{rad/s})^2(1.2\ \text{m}) = 3.1\ \text{m/s}^2.$$

Or we can solve instead: $a_R = v^2/r = (1.9\ \text{m/s})^2/(1.2\ \text{m}) = 3.0\ \text{m/s}^2$. (The difference is due to rounding off.) What force causes this acceleration? Is it a force of friction exerted by the merry-go-round?

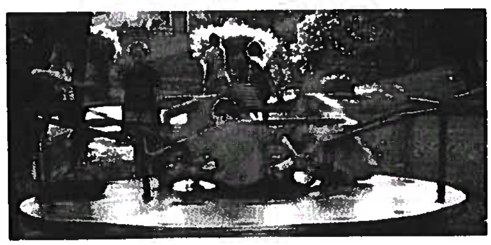


FIGURE 8–7 Merry-go-round.

PHYSICS APPLIED

Hard drive and bit speed

EXAMPLE 8–4 **Hard drive.** The platter of the hard disk of a computer rotates at 5400 rpm (revolutions per minute). (*a*) What is the angular velocity of the disk? (*b*) If the reading head of the drive is located 3.0 cm from the rotation axis, what is the speed of the disk below it? (*c*) What is the linear acceleration of this point? (*d*) If a single bit requires $5\,\mu\text{m}$ of length along the motion direction, how many bits per second can the writing head write when it is 3.0 cm from the axis?

SOLUTION (*a*) The angular velocity

$$\omega = 2\pi f = (2\pi\,\text{rad/rev})\frac{(5400\,\text{rev/min})}{(60\,\text{s/min})} = 570\,\text{rad/s}.$$

(*b*) The speed of a point 3.0 cm out from the axis is

$$v = r\omega = (3.0 \times 10^{-2}\,\text{m})(570\,\text{rad/s}) = 17\,\text{m/s}.$$

(*c*) The linear acceleration has two components, tangential and radial. Since ω = constant, then $\alpha = 0$, so $a_{\text{tan}} = r\alpha = 0$. The radial acceleration is

$$a_{\text{R}} = \omega^2 r = (570\,\text{rad/s})^2(0.030\,\text{m}) = 9700\,\text{m/s}^2$$

toward the axis.

(*d*) Each bit requires 5.0×10^{-6} m, so at a speed of 17 m/s, the number of bits passing the head per second is

$$\frac{17\,\text{m/s}}{5.0 \times 10^{-6}\,\text{m}} = 3.4 \times 10^6 \text{ bits per second.}$$

EXAMPLE 8–5 **Centrifuge acceleration.** A centrifuge rotor is accelerated from rest to 20,000 rpm in 5.0 min. What is its average angular acceleration?

SOLUTION To calculate $\bar{\alpha} = \Delta\omega/\Delta t$ we need the initial and final angular velocities. The initial angular velocity is $\omega = 0$. The final angular velocity is

$$\omega = 2\pi f = (2\pi\,\text{rad/rev})\frac{(20{,}000\,\text{rev/min})}{(60\,\text{s/min})} = 2100\,\text{rad/s}.$$

Then, since $\bar{\alpha} = \Delta\omega/\Delta t$ and $\Delta t = 5.0\,\text{min} = 300\,\text{s}$, we have

$$\bar{\alpha} = \frac{\omega - \omega_0}{\Delta t} = \frac{2100\,\text{rad/s} - 0}{300\,\text{s}} = 7.0\,\text{rad/s}^2.$$

That is, every second the rotor's angular velocity increases by 7.0 rad/s, or by $(7.0/2\pi) = 1.1$ revolutions per second.

8–2 Kinematic Equations for Uniformly Accelerated Rotational Motion

In Chapter 2, we derived the important equations (2–10) that relate acceleration, velocity, and distance for the situation of uniform linear acceleration. Those equations were derived from the definitions of linear velocity and acceleration, assuming constant acceleration. The definitions of angular velocity and angular acceleration are the same as for their linear counterparts, except that θ has replaced the linear displacement x, ω has replaced v, and α has replaced a. Therefore, the angular equations for **constant angular acceleration** will be analogous to Eqs. 2–10 with x replaced by θ, v by ω, and a by α, and they can be derived in exactly the same way. We summarize them here, opposite their linear equivalents (we've chosen $x_0 = 0$, and $\theta_0 = 0$):

Angular	Linear		
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$	[constant α, a]	**(8–9a)**
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$	[constant α, a]	**(8–9b)**
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$	[constant α, a]	**(8–9c)**
$\overline{\omega} = \dfrac{\omega + \omega_0}{2}$	$\overline{v} = \dfrac{v + v_0}{2}$	[constant α, a]	**(8–9d)**

Uniformly accelerated rotational motion

Note that ω_0 represents the angular velocity at $t = 0$, whereas θ and ω represent the angular position and velocity, respectively, at time t. Since the angular acceleration is constant, $\alpha = \overline{\alpha}$. These equations are of course also valid for constant angular velocity, for which case $\alpha = 0$ and we have $\omega = \omega_0$, $\theta = \omega_0 t$, and $\overline{\omega} = \omega$.

EXAMPLE 8–6 **Centrifuge revisited.** Through how many revolutions has the centrifuge rotor of Example 8–5 turned during its acceleration period? Assume constant angular acceleration.

SOLUTION We know that $\omega_0 = 0$, $\omega = 2100\,\text{rad/s}$, $\alpha = \overline{\alpha} = 7.0\,\text{rad/s}^2$, and $t = 300\,\text{s}$. We could use either Eq. 8–9b or 8–9c to find θ. The former gives

$$\theta = 0 + \tfrac{1}{2}(7.0\,\text{rad/s}^2)(300\,\text{s})^2 = 3.15 \times 10^5\,\text{rad},$$

where we have kept an extra digit because this is an intermediate result. To find the total number of revolutions, we divide by 2π and obtain

$$\frac{3.15 \times 10^5\,\text{rad}}{2\pi\,\text{rad/rev}} = 5.0 \times 10^4\,\text{rev}.$$

8–3 Rolling Motion

The rolling motion of a ball or wheel is familiar in everyday life: a ball rolling across the floor, or the wheels and tires of a car or bicycle rolling along the pavement. Rolling *without slipping* is readily analyzed and depends on static friction between the rolling object and the ground. The friction is static because the rolling object's point of contact with the ground is at rest at each moment. (Kinetic friction comes in if, for example, you brake too hard so the tires skid, or you accelerate so fast that you "burn rubber"—but these are more complicated situations.)

Rolling without slipping involves both rotation and translation. But there is a simple relation between the linear speed v of the axle and the

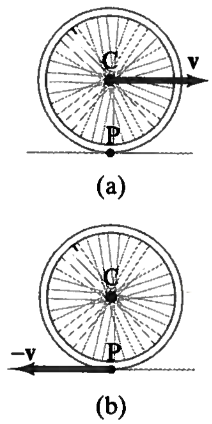
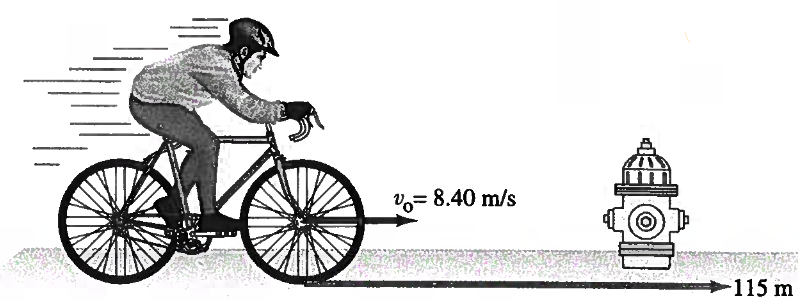


FIGURE 8–8 (a) A wheel rolling to the right. Its center C moves with velocity **v**. (b) The same wheel as seen from a reference frame in which the axle of the wheel C is at rest—that is, we are moving to the right with velocity **v** relative to part (a). Point P, which was at rest in (a), here in (b) moves to the left with velocity $-\mathbf{v}$ as shown. (See also Section 3–8 on relative velocity.

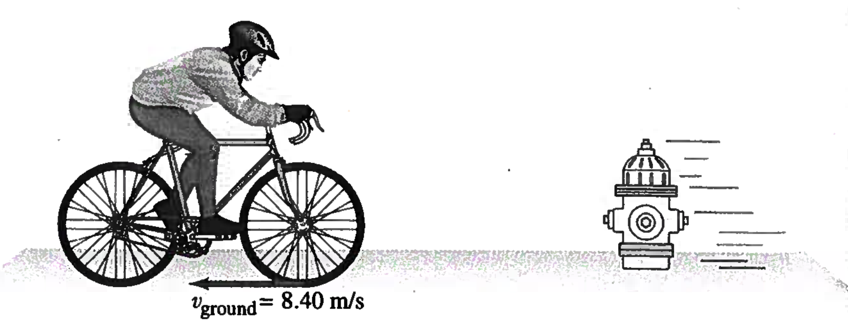
angular velocity ω of the rotating wheel or sphere, namely $v = r\omega$ where r is radius, as we now show. Figure 8–8a shows a wheel rolling to the right without slipping. At the moment shown, point P on the wheel is in contact with the ground and is momentarily at rest. The velocity of the axle at the wheel's center C is **v**. In Figure 8–8b we have put ourselves in the reference frame of the wheel—that is, we are moving to the right with velocity **v** relative to the ground. In this reference frame the axle C is at rest, whereas the ground and point P are moving to the left with velocity $-\mathbf{v}$ as shown. Here we are seeing pure rotation. So we can use Eq. 8–4 to obtain $v = r\omega$ where r is the radius of the wheel, and v is still equal to the translational speed of the wheel's center.

EXAMPLE 8–7 **Bicycle.** A bicycle slows down uniformly from $v_0 = 8.40\,\mathrm{m/s}$ to rest over a distance of 115 m, Fig. 8–9a. Each wheel and tire has an overall diameter of 68.0 cm. Determine (*a*) the angular velocity of the wheels at the initial instant, (*b*) the total number of revolutions each wheel rotates in coming to rest, (*c*) the angular acceleration of the wheel, and (*d*) the time it took to come to a stop.

SOLUTION (*a*) Let us put ourselves in the reference frame of the bike—that is, as if we were riding the bike. Then the ground is going past us, initially, at a speed of 8.40 m/s, Fig. 8–9b. Since the tire is in contact with the ground at any moment, then a point on the rim of the tire (such as that touching the ground) moves at an initial speed of $v = 8.40\,\mathrm{m/s}$ in



(a) Bike as seen from the ground ($t = 0$).



(b) From rider's reference frame, the ground is moving to the rear at an initial speed of 8.40 m/s ($t = 0$).

FIGURE 8–9 Example 8–7.

this reference frame. Hence the initial angular velocity of the wheel is

$$\omega_0 = \frac{v_0}{r} = \frac{8.40\text{ m/s}}{0.340\text{ m}} = 24.7\text{ rad/s}.$$

(*b*) In coming to a stop, 115 m of ground passes beneath the tire. Because the tire is in firm contact with the ground, any point on the edge of the rotating tire travels 115 m total. Each revolution corresponds to a distance of $2\pi r$, so the number of revolutions the wheel makes in coming to a stop is

$$\frac{115\text{ m}}{2\pi r} = \frac{115\text{ m}}{(2\pi)(0.340\text{ m})} = 53.8\text{ rev}.$$

(*c*) The angular acceleration of the wheel can be obtained from Eq. 8–9c:

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta} = \frac{0 - (24.7\text{ rad/s})^2}{2(2\pi)(53.8\text{ rev})} = -0.902\text{ rad/s}^2,$$

where we have set $\theta = 2\pi\text{ rad/rev} \times 53.8\text{ rev}$ ($= 338$ rad) because each revolution corresponds to 2π radians. [Alternatively, we could have used Eq. 8–1 to get the total θ: $\theta = l/r = 115\text{ m}/0.340\text{ m} = 338\text{ rad}$.]

(*d*) Eq. 8–9a or b allows us to solve for the time. The first is easier:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 24.7\text{ rad/s}}{-0.902\text{ rad/s}^2} = 27.4\text{ s}.$$

8–4 Torque

We have so far discussed rotational kinematics—the description of rotational motion in terms of angle, angular velocity, and angular acceleration. Now we discuss the dynamics, or causes, of rotational motion. Just as we found analogies between linear and rotational motion for the description of motion, so rotational equivalents for dynamics exist as well.

To make an object start rotating about an axis clearly requires a force. But the direction of this force, and where it is applied, are also important. Take, for example, an ordinary situation such as the door in Fig. 8–10 (looking down from above). If you apply a force $\mathbf{F}_1$ perpendicular to the door as shown, you will find that the greater the magnitude, F_1, the more quickly the door opens. (We assume that only this one force acts—we ignore friction in the hinges, and so on.) But now if you apply the same magnitude force at a point closer to the hinge, say $\mathbf{F}_2$ in Fig. 8–10, you will find that the door will not open so quickly. The effect of the force is less. Indeed, it is found that the angular acceleration of the door is proportional not only to the magnitude of the force, but it is also directly proportional to *the perpendicular distance from the axis of rotation to the line along which the force acts.* This distance is called the **lever arm**, or **moment arm**, of the force, and is labeled r_1 and r_2 for the two forces in Fig. 8–10. Thus, if r_1 in Fig. 8–10 is three times as large as r_2, then the angular acceleration of the door will be three times as great, assuming that the magnitudes of the forces are the same. To say it another way, if $r_1 = 3r_2$, then F_2 must be three times as large as F_1 to give the same angular acceleration. (Figure 8–11 shows two examples of tools whose long lever arms help produce large torque.)

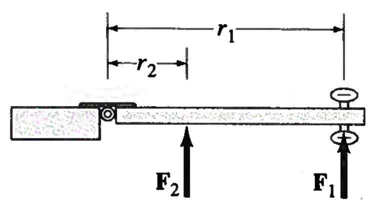
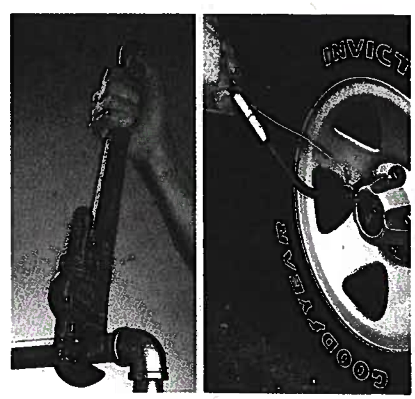


FIGURE 8–10 Applying the same force with different lever arms, r_1 and r_2. If $r_1 = 3r_2$, then to create the same effect (angular acceleration), F_2 needs to be three times F_1, or $F_1 = \frac{1}{3}F_2$.

FIGURE 8–11 (a) A plumber can exert greater torque using a wrench with a long lever arm. (b) A tire iron too can have a long lever arm.



(a) (b)

Torque defined

The angular acceleration, then, is proportional to the product of the *force times the lever arm.* This product is called the *moment of the force* about the axis, or, more commonly, it is called the **torque**, and is abbreviated τ (Greek lowercase letter tau). Thus, the angular acceleration α of an object is directly proportional to the net applied torque, τ:

$$\alpha \propto \tau.$$

This is the rotational analog of Newton's second law for linear motion, $a \propto F$.

We defined the lever arm as the *perpendicular* distance of the axis of rotation to the line of action of the force—that is, the distance which is perpendicular both to the axis of rotation and to an imaginary line drawn along the direction of the force. We do this to take into account the effect of forces acting at an angle. It is clear that a force applied at an angle, such as $\mathbf{F}_3$ in Fig. 8–12, will be less effective than the same magnitude force applied straight on, such as $\mathbf{F}_1$ (Fig. 8–12a). And if you push on the end of the door so that the force is directed at the hinge (the axis of rotation), as indicated by $\mathbf{F}_4$, the door will not rotate at all.

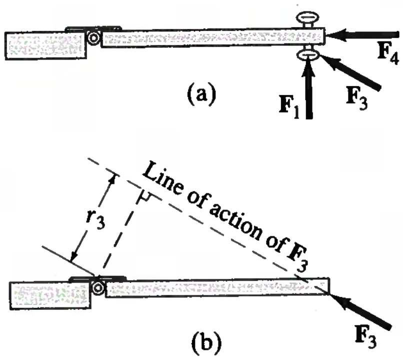


FIGURE 8–12 (a) Forces acting at different angles at the doorknob. (b) The lever arm is defined as the perpendicular distance from the axis of rotation (the hinge) to the line of action of the force.

The lever arm for a force such as $\mathbf{F}_3$ is found by drawing a line along the direction of $\mathbf{F}_3$ (this is the "line of action" of $\mathbf{F}_3$). Then we draw another line, perpendicular to this "line of action", that goes to the axis and is perpendicular to it. The length of this second line is the lever arm for $\mathbf{F}_3$ and is labeled r_3 in Fig. 8–12b. The lever arm is perpendicular both to the line of action of the force and, at its other end, perpendicular to the rotation axis.

The magnitude of the torque associated with $\mathbf{F}_3$ is then r_3F_3. This short lever arm and the corresponding smaller torque associated with $\mathbf{F}_3$ is consistent with the observation that $\mathbf{F}_3$ is less effective in accelerating the door than is $\mathbf{F}_1$. When the lever arm is defined in this way, experiment shows that the relation $\alpha \propto \tau$ is valid in general. Notice in Fig. 8–12 that the line of action of the force $\mathbf{F}_4$ passes through the hinge and hence its lever arm is zero. Consequently, zero torque is associated with $\mathbf{F}_4$ and it gives rise to no angular acceleration, in accord with everyday experience.

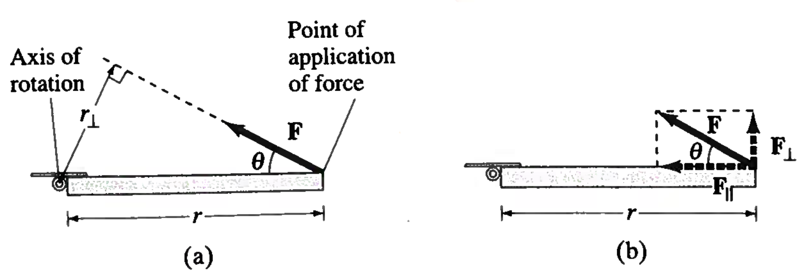
In general, then, we can write the torque about a given axis as

$$\tau = r_\perp F, \qquad \text{(8–10a)}$$

where $r_\perp$ is the lever arm, and the perpendicular symbol ($\perp$) reminds us that we must use the distance from the axis of rotation that is perpendicular to the line of action of the force (Fig. 8–13a).

An alternate but equivalent way of determining the torque associated with a force is to resolve the force into components parallel and perpendicular to the line joining the point of application of the force to the axis, as shown in Fig. 8–13b. The component $F_\parallel$ exerts no torque since it is directed

FIGURE 8–13
Torque $= r_\perp F = rF_\perp$



at the rotation axis (its moment arm is zero). Hence the torque will be equal to $F_\perp$ times the distance r from the axis to the point of application of the force:

$$\tau = rF_\perp . \qquad \textbf{(8–10b)}$$

That this gives the same result as Eq. 8–10a can be seen from the fact that $F_\perp = F\sin\theta$ and $r_\perp = r\sin\theta$. [Note that θ is the angle between the directions of **F** and r (radial line from the axis to where **F** acts)]. So

$$\tau = rF\sin\theta \qquad \textbf{(8–10c)}$$

Magnitude of a torque

in either case. We can use any of Eqs. 8–10 to calculate the torque, whichever is easiest.

Since torque is a distance times a force, it is measured in units of m·N in SI units,[†] cm·dyne in the cgs system, and ft·lb in the English system.

EXAMPLE 8–8 **Biceps torque.** The biceps muscle exerts a vertical force on the lower arm as shown in Figs. 8–14a and b. For each case, calculate the torque about the axis of rotation through the elbow joint, assuming the muscle is attached 5.0 cm from the elbow as shown.

SOLUTION (*a*) $F = 700\,\text{N}$ and $r_\perp = 0.050\,\text{m}$, so

$$\tau = r_\perp F = (0.050\,\text{m})(700\,\text{N}) = 35\,\text{m·N}.$$

(*b*) Because the arm is at an angle, the lever arm is shorter (Fig. 8–14c): $r_\perp = (0.050\,\text{m})(\sin 60°)$. F is still 700 N, so

$$\tau = (0.050\,\text{m})(0.866)(700\,\text{N}) = 30\,\text{m·N}.$$

The arm can exert less torque at this angle. Weight machines at gyms are often designed to take this variation with angle into account.

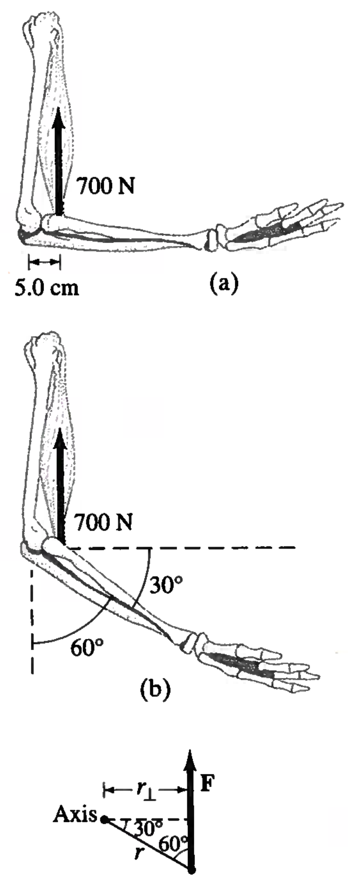


FIGURE 8–14 Example 8–8.

CONCEPTUAL EXAMPLE 8–9 **The chimp is no wimp.** Adult chimpanzees have muscle mass of only about one third of an adult human male, but have been shown to be over twice as strong in some movements. Can you account for this?

RESPONSE The performance difference can be traced to anatomy. The attachment point of the biceps muscle to the forearm, for example, is much farther away from the elbow in chimps than in humans. The increase in the lever arm means that the same muscle force exerted by chimpanzee biceps produces a greater torque. A chimp has more leverage.

When more than one torque acts on a body, the acceleration α is found to be proportional to the *net* torque. If all the torques acting on a body tend to rotate it in the same direction, the net torque is the sum of the torques. But if, say, one torque acts to rotate a body in one direction,

[†]Note that the units for torque are the same as those for energy. We write the unit for torque here as m·N (in SI) to help distinguish it from energy (N·m) because the two quantities are very different. An obvious difference is that energy is a scalar, whereas torque has a direction and is a vector. The special name *joule* (1 J = 1 N·m) is used only for energy (and for work), *never* for torque.

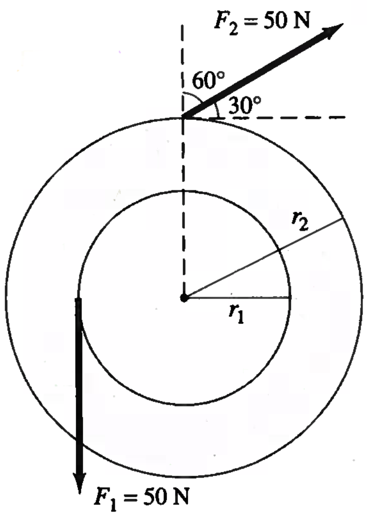
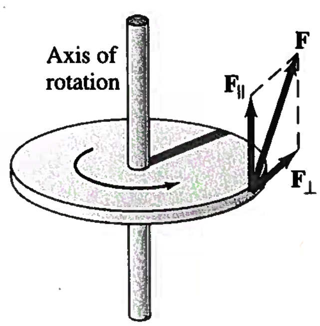


FIGURE 8–15 Example 8–10. The torque due to $\mathbf{F}_1$ tends to accelerate the wheel counterclockwise, whereas the torque due to $\mathbf{F}_2$ tends to accelerate the wheel clockwise.

FIGURE 8–16 Only the component of **F** that acts in the plane perpendicular to the rotation axis, $\mathbf{F}_\perp$, acts to turn the wheel about the axis. The component parallel to the axis, $\mathbf{F}_\parallel$, would tend to move the axis itself, which we assume is fixed.



and a second torque acts to rotate the body in the opposite direction (as in Fig. 8–15), the net torque is the difference of the two torques. We can assign a positive sign to torques that act to rotate the body in one direction (say counterclockwise) and a negative sign to torques that act to rotate the body in the opposite direction (clockwise).

EXAMPLE 8–10 **Torque on a compound wheel.** Two thin cylindrical wheels, of radii $r_1 = 30$ cm and $r_2 = 50$ cm, are attached to each other on an axle that passes through the center of each, as shown in Fig. 8–15. Calculate the net torque on this compound wheel due to the two forces shown, each of magnitude 50 N.

SOLUTION The force $\mathbf{F}_1$ acts to rotate the system counterclockwise, whereas $\mathbf{F}_2$ acts to rotate it clockwise. So the two forces act in opposition to each other. We must choose one direction of rotation to be positive—say, counterclockwise. Then $\mathbf{F}_1$ exerts a positive torque, $\tau_1 = r_1 F_1$, since the lever arm is r_1. $\mathbf{F}_2$, on the other hand, produces a negative (clockwise) torque and does not act perpendicular to r_2, so we must use its perpendicular component to calculate the torque it produces: $\tau_2 = -r_2 F_{2\perp} = -r_2 F_2 \sin\theta$, where $\theta = 60°$. (Note that θ must be the angle between $\mathbf{F}_2$ and a radial line from the axis.) Hence the net torque is

$$\begin{aligned}\tau &= r_1 F_1 - r_2 F_2 \sin 60° \\ &= (0.30\,\text{m})(50\,\text{N}) - (0.50\,\text{m})(50\,\text{N})(0.866) = -6.7\,\text{m}\cdot\text{N}.\end{aligned}$$

This net torque acts to accelerate the rotation of the wheel in the clockwise direction. Note that the two forces have the same magnitude, yet produce a net torque because their lever arms are different.

[Since we are interested only in rotation about a fixed axis, we consider only forces that act in a plane perpendicular to the axis of rotation. If there is a force (or component of a force) acting parallel to the axis of rotation, it will tend to turn the axis of rotation—the component $\mathbf{F}_\parallel$ in Fig. 8–16 is an example. Since we are assuming the axis remains fixed in direction, either there can be no such forces or else the axis must be an axle (or the like) which is mounted in bearings or hinges that hold the axis fixed. Thus, only a force, or component of a force ($\mathbf{F}_\perp$ in Fig. 8–16), in a plane perpendicular to the axis will give rise to rotation about the axis, and it is only these that we consider.]

8–5 Rotational Dynamics; Torque and Rotational Inertia

We have discussed that the angular acceleration α of a rotating body is proportional to the net torque τ applied to it:

$$\alpha \propto \Sigma\tau$$

where we write $\Sigma\tau$ to remind us† that it is the net torque (sum of all torques acting on the body) that is proportional to α. This corresponds to Newton's second law for translational motion, $a \propto \Sigma F$, but here torque has taken the place of force, and, correspondingly, the angular accelera-

†Recall from Chapter 4 that Σ (Greek letter sigma) means "sum of."

tion α takes the place of the linear acceleration a. In the linear case, the acceleration is not only proportional to the net force, but it is also inversely proportional to the inertia of the body, which we call its mass, m. Thus we could write $a = \Sigma F/m$. But what plays the role of mass for the rotational case? That is what we now set out to determine. At the same time, we will see that the relation $\alpha \propto \Sigma\tau$ follows directly from Newton's second law, $\Sigma F = ma$.

We first consider a very simple case: a particle of mass m rotating in a circle of radius r at the end of a string or rod whose mass we can ignore (Fig. 8–17), and we assume a single force F acts on it as shown. The torque that gives rise to the angular acceleration is $\tau = rF$. If we make use of Newton's second law for linear quantities, $\Sigma F = ma$, and Eq. 8–5 relating the angular acceleration to the tangential linear acceleration, $a_{tan} = r\alpha$, we have

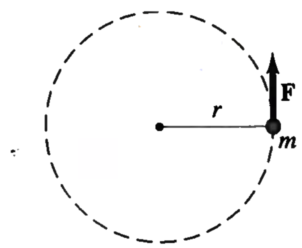


FIGURE 8–17 A mass m rotating in a circle of radius r about a fixed point.

$$F = ma$$
$$= mr\alpha.$$

When we multiply both sides by r, we find that the torque $\tau = rF$ is given by

$$\tau = mr^2\alpha. \qquad \text{[single particle]} \qquad \textbf{(8–11)}$$

Here at last we have a direct relation between the angular acceleration and the applied torque τ. The quantity mr^2 represents the *rotational inertia* of the particle and is called its *moment of inertia*.

Now let us consider a rotating rigid body, such as a wheel rotating about an axis through its center, such as an axle. We can think of the wheel as consisting of many particles located at various distances from the axis of rotation. We can apply Eq. 8–11 to each particle of the body, and then sum over all the particles. The sum of the various torques is just the total torque, $\Sigma\tau$, so we obtain:

$$\Sigma\tau = (\Sigma\, mr^2)\alpha \qquad \textbf{(8–12)}$$

where we factored out the α since it is the same for all the particles of the body. The sum, Σmr^2, represents the sum of the masses of each particle in the body multiplied by the square of the distance of that particle from the axis of rotation. If we give each particle a number (1, 2, 3, ...), then $\Sigma mr^2 = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots$. This quantity is called the **moment of inertia** (or *rotational inertia*) of the body, I:

Moment of inertia

$$I = \Sigma mr^2 = m_1r_1^2 + m_2r_2^2 + \cdots. \qquad \textbf{(8–13)}$$

Combining Eqs. 8–12 and 8–13, we can write

NEWTON'S SECOND LAW FOR ROTATION

$$\Sigma\tau = I\alpha. \qquad \textbf{(8–14)}$$

This is the rotational equivalent of Newton's second law. It is valid for the rotation of a rigid body about a fixed axis.†

We see that the moment of inertia, I, which is a measure of the rotational inertia of a body, plays the same role for rotational motion that mass does for translational motion. As can be seen from Eq. 8–13, the rotational inertia of an object depends not only on its mass, but also on how that mass is distributed with respect to the axis. For example, a large-diameter

†It can be shown that Eq. 8–14 is valid also when the body is translating with acceleration, as long as I and α are calculated about the center of mass of the body, and the rotation axis through the CM doesn't change direction.

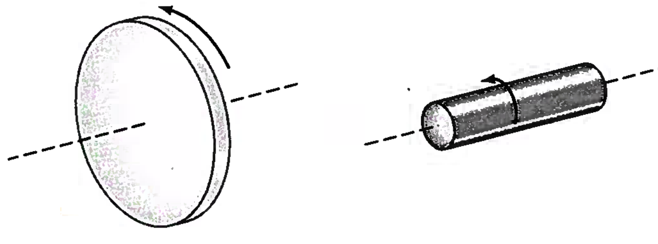


FIGURE 8–18 A large-diameter cylinder has greater rotational inertia than one of smaller diameter but equal mass.

Careful: Mass can not *be considered concentrated at* CM *for rotational motion*

cylinder will have greater rotational inertia than one of equal mass but smaller diameter (and therefore greater length), Fig. 8–18. The former will be harder to start rotating, and harder to stop. When the mass is concentrated farther from the axis of rotation, the rotational inertia is greater. For rotational motion, the mass of a body *cannot* be considered as concentrated at its center of mass.

8–6 Solving Problems in Rotational Dynamics

Whenever you use Eq. 8–14, remember to use a consistent set of units, which in SI is: α in rad/s^2; τ in m·N; and the moment of inertia, I, in kg·m^2.

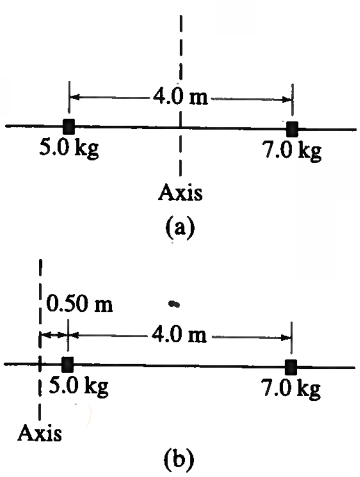


FIGURE 8–19 Example 8–11: calculating the moment of inertia.

EXAMPLE 8–11 **Two weights on a bar: different axis, different *I*.** Two "weights" of mass 5.0 kg and 7.0 kg are mounted 4.0 m apart on a light rod (whose mass can be ignored), as shown in Fig. 8–19. Calculate the moment of inertia of the system (*a*) when rotated about an axis halfway between the weights, Fig. 8–19a, and (*b*) when the system rotates about an axis 0.50 m to the left of the 5.0-kg mass (Fig. 8–19b).

SOLUTION (*a*) Both weights are the same distance, 2.0 m, from the axis of rotation. Thus

$$I = \Sigma mr^2 = (5.0\ \text{kg})(2.0\ \text{m})^2 + (7.0\ \text{kg})(2.0\ \text{m})^2$$
$$= 20\ \text{kg}\cdot\text{m}^2 + 28\ \text{kg}\cdot\text{m}^2 = 48\ \text{kg}\cdot\text{m}^2.$$

(*b*) The 5.0-kg mass is now 0.50 m from the axis and the 7.0-kg mass is 4.50 m from the axis. Then

$$I = \Sigma mr^2 = (5.0\ \text{kg})(0.50\ \text{m})^2 + (7.0\ \text{kg})(4.5\ \text{m})^2$$
$$= 1.3\ \text{kg}\cdot\text{m}^2 + 142\ \text{kg}\cdot\text{m}^2 = 143\ \text{kg}\cdot\text{m}^2.$$

I depends on axis of rotation and on distribution of mass

The above Example illustrates two important points. First, the moment of inertia of a given system is different for different axes of rotation. Second, we see in part (*b*) that mass close to the axis of rotation contributes little to the total moment of inertia; in this example, the 5.0-kg object contributed less than 1 percent to the total.

For most ordinary bodies, the mass is distributed continuously, and the calculation of the moment of inertia, $\Sigma\ mr^2$, can be difficult. Expressions can, however, be worked out (using calculus) for the moments of inertia of regularly shaped bodies in terms of their dimensions. Figure 8–20 gives these expressions for a number of solids rotated about the axes specified. The only one for which the result is obvious is that for the thin hoop or ring rotated about an axis passing through its center perpendicular to the plane of the hoop (Fig. 8–20a). For this object, all the mass is concentrated at the same distance from the axis, R. Thus $\Sigma mr^2 = (\Sigma m)R^2 = MR^2$, where M is the total mass of the hoop.

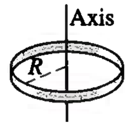
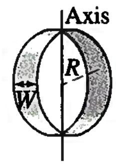
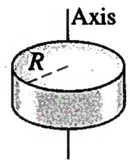
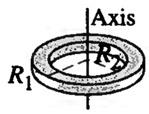
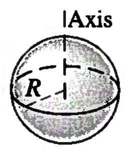
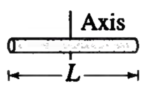
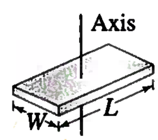
	Object	Location of axis		Moment of inertia
(a)	Thin hoop of radius R	Through center		MR^2
(b)	Thin hoop of radius R and width W	Through central diameter		$\frac{1}{2}MR^2 + \frac{1}{12}MW^2$
(c)	Solid cylinder of radius R	Through center		$\frac{1}{2}MR^2$
(d)	Hollow cylinder of inner radius R_1 and outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	Uniform sphere of radius R	Through center		$\frac{2}{5}MR^2$
(f)	Long uniform rod of length L	Through center		$\frac{1}{12}ML^2$
(g)	Long uniform rod of length L	Through end		$\frac{1}{3}ML^2$
(h)	Rectangular thin plate, of length L and width W	Through center		$\frac{1}{12}M(L^2 + W^2)$

FIGURE 8–20 Moments of inertia for various objects of uniform composition.

PROBLEM SOLVING Rotational Motion

1. As always, draw a clear and complete diagram.
2. Draw a free-body diagram for the body under consideration (or for each body if more than one), showing only (and all) the forces acting on that body and exactly where they act, so you can determine the torque due to each. Gravity acts at the CG of the body (Section 7–8).
3. Identify the axis of rotation and calculate the torques about it. Choose positive and negative directions of rotation (counterclockwise and clockwise), and assign the correct sign to each torque.
4. Apply Newton's second law for rotation, $\Sigma\tau = I\alpha$. If the moment of inertia is not given, and it is not the unknown sought, you need to determine it first. Use consistent units, which in SI are: α in rad/s^2; τ in m·N; and I in kg·m^2.
5. Solve the resulting equation(s) for the unknown(s).
6. As always, do a rough estimate to determine if your answer is reasonable: does it make sense?

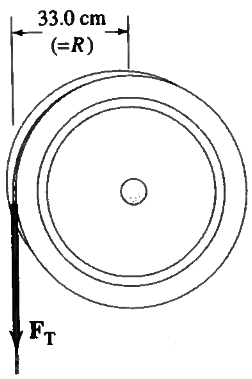


FIGURE 8–21 Example 8–12.

EXAMPLE 8–12 **A heavy pulley.** A 15.0-N force (represented by $\mathbf{F}_T$) is applied to a cord wrapped around a pulley of mass $M = 4.00$ kg and radius $R = 33.0$ cm, Fig. 8–21. The pulley is observed to accelerate uniformly from rest to reach an angular speed of 30.0 rad/s in 3.00 s. If there is a frictional torque (at the axle), $\tau_{fr} = 1.10$ m·N, determine the moment of inertia of the pulley. The pulley is assumed to rotate about its center. Its free-body diagram is shown in Fig. 8–21, although the friction force is not shown since we are given only its torque.

SOLUTION We can calculate the moment of inertia from Eq. 8–14, $\Sigma\tau = I\alpha$, since from the measurements given we can determine $\Sigma\tau$ and α. The net torque is the applied torque due to $\mathbf{F}_T$ minus the frictional torque; we take positive to be counterclockwise:

$$\Sigma\tau = (0.330\text{ m})(15.0\text{ N}) - 1.10\text{ m}\cdot\text{N} = 3.85\text{ m}\cdot\text{N}.$$

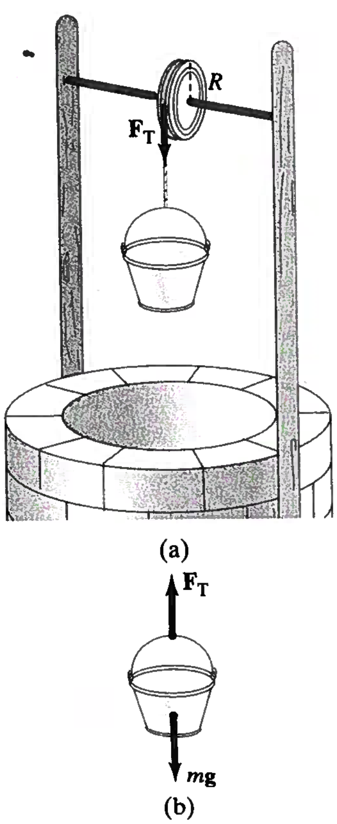
The angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{30.0\text{ rad/s} - 0}{3.00\text{ s}} = 10.0\text{ rad/s}^2.$$

Hence

$$I = \frac{\Sigma\tau}{\alpha} = \frac{3.85\text{ m}\cdot\text{N}}{10.0\text{ rads/s}^2} = 0.385\text{ kg}\cdot\text{m}^2.$$

FIGURE 8–22 Example 8–13, with free-body diagram for the falling bucket of mass m shown in (b).



EXAMPLE 8–13 **Pulley and bucket: raising water from a well.** Consider again the pulley in Fig. 8–21. But this time, suppose that, instead of a constant 15.0-N force being exerted on the cord, we now have a bucket of weight 15.0 N (mass $m = 1.53$ kg) hanging from the cord, which we assume not to stretch or slip on the pulley. See Fig. 8–22a. (*a*) Calculate the angular acceleration α of the pulley and the linear acceleration a of the bucket. (*b*) Determine the angular velocity ω of the pulley and the linear velocity v of the bucket at $t = 3.00$ s if the pulley (and bucket) start from rest at $t = 0$.

SOLUTION (*a*) Let F_T be the tension in the cord. Then a force F_T acts at the edge of the pulley, and we have for the rotation of the pulley:

$$I\alpha = \Sigma\tau = F_T R - \tau_{fr}. \qquad \text{[pulley]}$$

Next we look at the (linear) motion of the bucket of mass m. Figure 8–22b shows a free-body diagram for the bucket. Two forces act on the bucket: the force of gravity mg acts downward, and the tension of the cord F_T pulls upward. So by $\Sigma F = ma$, for the bucket we have (taking downward as positive):

$$mg - F_T = ma. \qquad \text{[bucket]}$$

Note that the tension F_T, which is the force exerted on the edge of the pulley, is *not* equal to the weight of the bucket ($= mg = 15.0$ N). There must be a net force (so $F_T < mg$) if the bucket is accelerating. Indeed, by the last equation above, $F_T = mg - ma$. To obtain α, we use Eq. 8–5,

$$a = R\alpha,$$

which is valid since the tangential acceleration of a point on the edge of the pulley is the same as the acceleration of the bucket if the cord doesn't stretch or slip. Substituting $F_T = mg - ma$ into the first equation above, we obtain

$$I\alpha = \Sigma\tau = F_T R - \tau_{fr} = (mg - mR\alpha)R - \tau_{fr} = mgR - mR^2\alpha - \tau_{fr}.$$

Now α appears on both sides of this last relation, so we solve for α:

$$\alpha(I + mR^2) = mgR - \tau_{fr},$$

and then

$$\alpha = \frac{mgR - \tau_{fr}}{I + mR^2}.$$

Then, since $I = 0.385\,\mathrm{kg\cdot m^2}$ (Example 8–12),

$$\alpha = \frac{(15.0\,\mathrm{N})(0.330\,\mathrm{m}) - 1.10\,\mathrm{m\cdot N}}{0.385\,\mathrm{kg\cdot m^2} + (1.53\,\mathrm{kg})(0.330\,\mathrm{m})^2} = 6.98\,\mathrm{rad/s^2}.$$

The angular acceleration is somewhat less in this case than the $10.0\,\mathrm{rad/s^2}$ of Example 8–12. Why? Because F_T ($= mg - ma$) is somewhat less than the weight of the bucket, mg. The linear acceleration of the bucket is

$$a = R\alpha = (0.330\,\mathrm{m})(6.98\,\mathrm{rad/s^2}) = 2.30\,\mathrm{m/s^2}.$$

(*b*) Since the angular acceleration is constant,

$$\omega = \omega_0 + \alpha t = 0 + (6.98\,\mathrm{rad/s^2})(3.00\,\mathrm{s}) = 20.9\,\mathrm{rad/s}$$

after 3.00 s. The linear velocity of the bucket is the same as that of a point on the wheel's edge:

$$v = R\omega = (0.330\,\mathrm{m})(20.9\,\mathrm{rad/s}) = 6.90\,\mathrm{m/s}.$$

The same result can also be obtained by using the linear equation $v = v_0 + at = 0 + (2.30\,\mathrm{m/s^2})(3.00\,\mathrm{s}) = 6.90\,\mathrm{m/s}$.

8–7 Rotational Kinetic Energy

The quantity $\frac{1}{2}mv^2$ is the kinetic energy of a body undergoing translational motion. A body rotating about an axis is said to have **rotational kinetic energy**. By analogy with translational KE, we would expect this to be given by the expression $\frac{1}{2}I\omega^2$ where I is the moment of inertia of the body and ω is its angular velocity. We can indeed show that this is true. Consider any rigid rotating object as made up of many tiny particles, each of mass m. If we let r represent the distance of any one particle from the axis of rotation, then its linear velocity is $v = r\omega$. The total kinetic energy of the whole body will be the sum of the kinetic energies of all its particles:

$$\begin{aligned}\mathrm{KE} &= \Sigma(\tfrac{1}{2}mv^2) = \Sigma(\tfrac{1}{2}mr^2\omega^2) \\ &= \tfrac{1}{2}\Sigma(mr^2)\omega^2,\end{aligned}$$

where we have factored out the $\frac{1}{2}$ and the ω^2 since they are the same for every particle of a rigid body. Since $\Sigma mr^2 = I$, the moment of inertia, we see that the kinetic energy of a rigid rotating object, as expected, is

$$\text{rotational KE} = \tfrac{1}{2}I\omega^2. \qquad \text{(8–15)}$$

Rotational KE

The units are joules, as with all other forms of energy.

An object that rotates while its center of mass (CM) undergoes translational motion will have both translational and rotational KE. Equation 8–15 gives the rotational KE if the rotation axis is fixed. If the object is moving (such as a wheel rolling down a hill), this equation is still valid as long as the rotation axis is fixed in direction. Then the total kinetic energy is

Total KE

$$\text{KE} = \tfrac{1}{2}Mv_{\text{CM}}^2 + \tfrac{1}{2}I_{\text{CM}}\omega^2,$$

where v_{CM} is the linear velocity of the CM, I_{CM} is the moment of inertia about an axis through the CM, ω is the angular velocity about this axis, and M is the total mass of the body.

EXAMPLE 8–14 **Sphere rolling down an incline.** What will be the speed of a solid sphere of mass M and radius R when it reaches the bottom of an incline if it starts from rest at a vertical height H and rolls without slipping? See Fig. 8–23. Ignore losses due to dissipative forces, and compare your result to that for an object *sliding* down a frictionless incline.

SOLUTION We use the law of conservation of energy, and we must now include rotational kinetic energy. The total energy at any point a vertical distance y above the base of the incline is

PROBLEM SOLVING

Rotational energy adds to other forms of energy to get the total energy which is conserved

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{\text{CM}}\omega^2 + Mgy,$$

where v is the speed of the CM. We equate the total energy at the top ($y = H$ and $v = \omega = 0$) to the total energy at the bottom ($y = 0$):

$$0 + 0 + MgH = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{\text{CM}}\omega^2 + 0.$$

From Fig. 8–20, the moment of inertia of a solid sphere about an axis through its CM is $I_{\text{CM}} = \tfrac{2}{5}MR^2$. Since the sphere rolls without slipping, the speed, v, of the center of mass with respect to the point of contact (which is momentarily at rest at any instant) is equal to the speed of a point on the edge relative to the center, as we saw in Section 8–3 (Fig. 8–8). We therefore have $\omega = v/R$. Hence

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}\left(\tfrac{2}{5}MR^2\right)\left(\frac{v^2}{R^2}\right) = MgH.$$

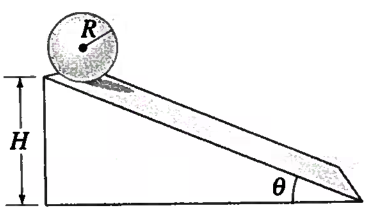


FIGURE 8–23 A sphere rolling down a hill has both translational and rotational kinetic energy. Example 8–14.

Canceling the M's and R's, we obtain

$$\left(\tfrac{1}{2} + \tfrac{1}{5}\right)v^2 = gH$$

or

$$v = \sqrt{\tfrac{10}{7}gH}.$$

Note first that v is independent of both the mass M and the radius R of the sphere. Also, we can compare this result for the speed of a rolling sphere to that for an object sliding down a plane without rotating and without friction (see Section 6–7, $\tfrac{1}{2}mv^2 = mgH$), in which case $v = \sqrt{2gH}$, which is greater. An object sliding without friction transforms its initial potential energy entirely into translational KE (none into rotational KE), so its speed is greater.

CONCEPTUAL EXAMPLE 8–15 **Who's fastest?** Several objects roll without slipping down an incline of vertical height H, all starting from rest at the same moment. The objects are a thin hoop (or a plain wedding band), a marble, a solid cylindrical battery (D-cell), an empty soup can, and an unopened soup can. In addition a greased box slides down without friction. In what order do they reach the bottom of the incline?

RESPONSE The sliding box wins every time! As we saw in Example 8–14, the speed of a rolling sphere at the bottom of the incline is less than that for a sliding box (without friction) because the potential energy loss (MgH) is transformed completely into translational KE for the box, whereas for rolling objects, the initial PE is shared between translational and rotational kinetic energy. For each of the rolling objects we can state that the loss in potential energy equals the increase in kinetic energy:

$$MgH = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{CM}\omega^2.$$

First we note that for all our rolling objects, the moment of inertia I_{CM} is a numerical factor times the mass M and the radius R^2 (Fig. 8–20). The mass M is in each term, so the translational speed v doesn't depend on M, nor does it depend on the radius R since $\omega = v/R$ so R^2 cancels out for all the rolling objects, just as in Example 8–14. Thus the speed v at the bottom depends only on that numerical factor in I_{CM} which expresses how the mass is distributed. Consequently, the hoop, with all its mass concentrated at radius R ($I_{CM} = MR^2$), will have the lowest speed and will arrive at the bottom behind the D-cell ($I_{CM} = \frac{1}{2}MR^2$) which in turn will be behind the marble ($I_{CM} = \frac{2}{5}MR^2$). The empty can, which is mainly a hoop plus a small disk, has its mass concentrated almost at R; so it will be a bit faster than the pure hoop but slower than the D-cell. See Fig. 8–24. The unopened soup can is much more complicated. It cannot be considered a solid cylinder because the soup can move about inside, and that will dissipate some energy; so we expect it to be slower than the D-cell, but that's about all we can safely say. For all the other objects, note that the speed at the bottom does not depend on the object's mass M or radius R, but only on its shape (and the height of the hill H).

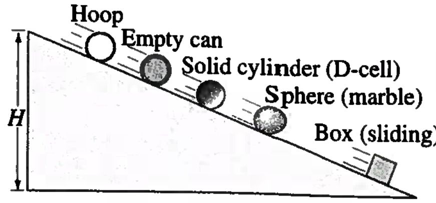


FIGURE 8–24 Example 8–15.

If there had been no friction between the sphere (and other rolling objects) and the plane in these Examples, the sphere would have slid rather than rolled. Friction must be present to make a round object roll. We did not need to take friction into account in the energy equation because it is *static* friction and does no work. If we assume the sphere is perfectly rigid and thus is in contact with the surface at a point, then the force of friction acts parallel to the plane. But the point of contact of the sphere at each instant does not slide—it moves perpendicular to the plane (first down and then up) as the sphere rolls (Fig. 8–25). Thus, no work is done by the friction force because the force and the motion are perpendicular. The reason the rolling objects in Examples 8–14 and 8–15 move down the slope more slowly than if they were sliding is *not* because friction is doing work. Rather it is because some of the gravitional PE is converted to rotational KE, leaving less for the translational KE.

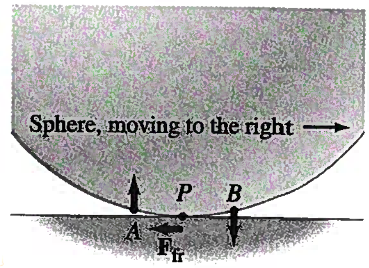


FIGURE 8–25 A sphere rolling to the right on a plane surface. The point in contact with the ground at any moment, point P, is momentarily at rest. Point A on the left of P is moving nearly vertically upward at the instant shown, and point B on the right is moving nearly vertically downward. (An instant later, point B will touch the plane and be at rest momentarily.)

The work done on a body rotating about a fixed axis, such as the wheel or pulley in Fig. 8–21 and 8–22, can be written using angular quantities. As shown in Fig. 8–26, a force F exerting a torque $\tau = rF$ on the wheel does work $W = F\,\Delta l$ in rotating the wheel a small distance Δl. The wheel has rotated through a small angle $\Delta\theta = \Delta l/r$ (see Eq. 8–1). Hence

$$W = F\,\Delta l = Fr\,\Delta\theta.$$

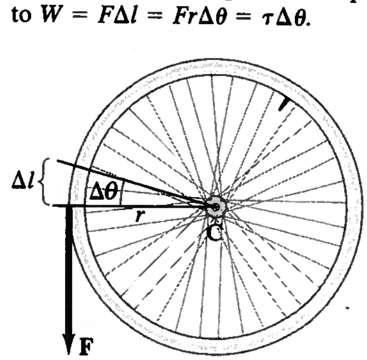
Since $\tau = rF$, then

$$W = \tau\,\Delta\theta \qquad \textbf{(8–16)}$$

Work done by torque

is the work done by the torque τ in rotating the wheel through an angle $\Delta\theta$.

FIGURE 8–26 Torque $\tau = rF$ does work in rotating a wheel equal to $W = F\Delta l = Fr\Delta\theta = \tau\Delta\theta$.



8–8 Angular Momentum and Its Conservation

Throughout this chapter we have seen that if we use the appropriate angular variables, the kinematic and dynamic equations for rotational motion are analogous to those for ordinary linear motion. In the last section (Section 8–7) we saw, for example, that rotational kinetic energy can be written as $\frac{1}{2}I\omega^2$, which is analogous to the translational KE $= \frac{1}{2}mv^2$. In like manner, the linear momentum, $p = mv$, has a rotational analog. It is called **angular momentum**, L, and for a body rotating about a fixed axis it is defined as

Angular momentum

$$L = I\omega, \tag{8–17}$$

where I is the moment of inertia, and ω is the angular velocity. The SI units for L are kg·m²/s.

We saw in Chapter 7 (Section 7–1) that Newton's second law can be written not only as $\Sigma F = ma$, but also more generally in terms of momentum (Eq. 7–2), $\Sigma F = \Delta p/\Delta t$. In a similar way, the rotational equivalent of Newton's second law, which we saw in Eq. 8–14 can be written as $\Sigma\tau = I\alpha$, can also be written in terms of angular momentum:

NEWTON'S SECOND LAW FOR ROTATION

$$\Sigma\tau = \frac{\Delta L}{\Delta t}, \tag{8–18}$$

where $\Sigma\tau$ is the net torque acting to rotate the body, and ΔL is the change in angular momentum in the time Δt. Equation 8–14, $\Sigma\tau = I\alpha$, is a special case of Eq. 8–18 when the moment of inertia is constant. This can be seen as follows. If a body has angular velocity ω_0 at time $t = 0$, and angular velocity ω at a time Δt later, then its angular acceleration (Eq. 8–3) is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{\Delta t}.$$

Then from Eq. 8–18, we have

$$\Sigma\tau = \frac{\Delta L}{\Delta t} = \frac{I\omega - I\omega_0}{\Delta t} = \frac{I(\omega - \omega_0)}{\Delta t} = I\frac{\Delta\omega}{\Delta t} = I\alpha,$$

which is Eq. 8–14.

Angular momentum is an important concept in physics because, under certain conditions, it is a conserved quantity. We can see from Eq. 8–18 that if the net torque $\Sigma\tau$ on a body is zero, then $\Delta L/\Delta t$ equals zero. That is, L does not change. This, then, is the **law of conservation of angular momentum** for a rotating body:

CONSERVATION OF ANGULAR MOMENTUM

> **The total angular momentum of a rotating body remains constant if the net torque acting on it is zero.**

The law of conservation of angular momentum is one of the great conservation laws of physics.

When there is zero net torque acting on a body, and the body is rotating about a fixed axis or about an axis through its CM such that its direction doesn't change, we can write

$$I\omega = I_0\omega_0 = \text{constant}.$$

I_0 and ω_0 are the moment of inertia and angular velocity, respectively, about that axis at some initial time ($t = 0$), and I and ω are their values at some other time. The parts of the body may alter their positions relative to one another, so that I changes. But then ω changes as well and the product $I\omega$ remains constant.

Many interesting phenomena can be understood on the basis of conservation of angular momentum. Consider a skater doing a spin on the tips of her skates, Fig. 8–27. She rotates at a relatively low speed when her arms are outstretched, but when she brings her arms in close to her body, she suddenly spins much faster. By remembering the definition of moment of inertia as $I = \Sigma mr^2$, it is clear that when she pulls her arms in closer to the axis of rotation, r is reduced for the arms so her moment of inertia is reduced. Since the angular momentum $I\omega$ remains constant (we ignore the small torque due to friction), if I decreases, then the angular velocity ω must increase. If the skater reduces her moment of inertia by a factor of 2, she will then rotate with twice the angular velocity.

A similar example is the diver shown in Fig. 8–28. The push as she leaves the board gives her an initial angular momentum about her CM. When she curls herself into the tuck position, she rotates quickly one or more times. She then stretches out again, increasing her moment of inertia, which reduces the angular velocity to a small value, and then she enters the water. The change in moment of inertia from the straight position to the tuck position can be a factor of as much as $3\frac{1}{2}$.

Note that for angular momentum to be conserved, the net torque must be zero, but the net force does not necessarily have to be zero. The net force on the diver in Fig. 8–28, for example, is not zero (gravity is acting), but the net torque on her is zero.

FIGURE 8–27 A skater doing a spin on ice, illustrating conservation of angular momentum: in (a), I is large and ω is small; in (b), I is smaller so ω is larger.

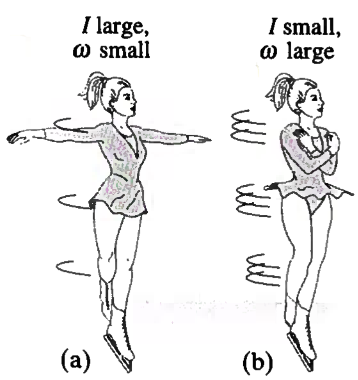
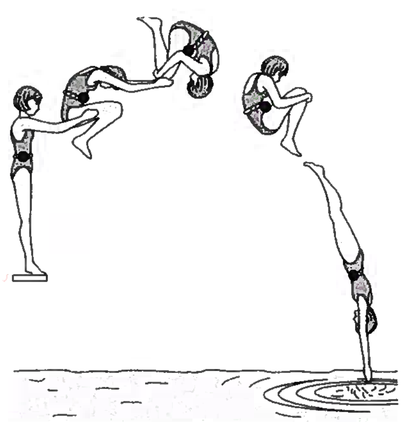


FIGURE 8–28 A diver rotates faster when arms and legs are tucked in than when they are outstretched. Angular momentum is conserved.



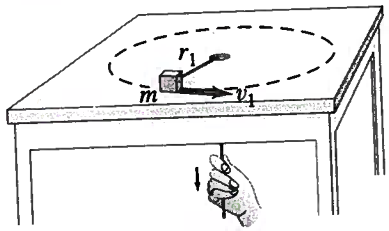


FIGURE 8–29 Example 8–16.

EXAMPLE 8–16 **Object rotating on a string of changing length.** A mass m attached to the end of a string revolves in a circle on a frictionless tabletop. The other end of the string passes through a hole in the table (Fig. 8–29). Initially, the mass revolves with a speed $v_1 = 2.4\,\text{m/s}$ in a circle of radius $r_1 = 0.80\,\text{m}$. The string is then pulled slowly through the hole so that the radius is reduced to $r_2 = 0.48\,\text{m}$. What is the speed, v_2, of the mass now?

SOLUTION The force exerted by the string on the mass m does not alter its angular momentum about the axis of rotation, because the force is exerted toward the axis so the lever arm is zero and $\tau = 0$. Hence, from conservation of angular momentum:

$$I_1\omega_1 = I_2\omega_2.$$

Our small mass is essentially a particle whose moment of inertia is $I = mr^2$ (Section 8–5, Eq. 8–11), so we have

$$mr_1^2\omega_1 = mr_2^2\omega_2,$$

or

$$\omega_2 = \omega_1\left(\frac{r_1^2}{r_2^2}\right).$$

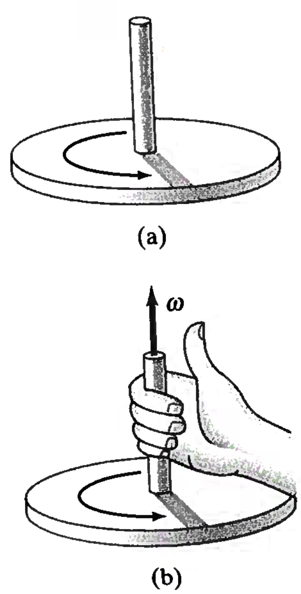
Then, since $v = r\omega$, we can write:

$$v_2 = r_2\omega_2 = r_2\omega_1\left(\frac{r_1^2}{r_2^2}\right) = r_2\frac{v_1}{r_1}\left(\frac{r_1^2}{r_2^2}\right) = v_1\frac{r_1}{r_2}$$

$$= (2.4\,\text{m/s})\left(\frac{0.80\,\text{m}}{0.48\,\text{m}}\right) = 4.0\,\text{m/s}.$$

* 8–9 Vector Nature of Angular Quantities

FIGURE 8–30 (a) Rotating wheel. (b) Right-hand rule for obtaining the direction of $\boldsymbol{\omega}$.



Up to now we have considered only the magnitudes of angular variables such as ω, α, and L. But they can be treated as vectors, and now we consider in what directions these vectors point. In fact, we have to *define* the directions for rotational quantities, and we take first the angular velocity, $\boldsymbol{\omega}$.

Consider the rotating wheel shown in Fig. 8–30a. The linear velocities of different particles of the wheel point in all different directions. The only unique direction in space associated with the rotation is along the axis of rotation, perpendicular to the actual motion. We therefore choose the axis of rotation to be the direction of the angular velocity vector, $\boldsymbol{\omega}$. Actually, there is still an ambiguity since $\boldsymbol{\omega}$ could point in either direction along the axis of rotation (up or down in Fig. 8–30a). The convention we use, called **the right-hand rule**, is the following: When the fingers of the right hand are curled around the rotation axis and point in the direction of the rotation, then the thumb points in the direction of $\boldsymbol{\omega}$. This is shown in Fig. 8–30b. Note that $\boldsymbol{\omega}$ points in the direction a right-handed screw would move when turned in the direction of rotation. Thus, if the rotation of the wheel in Fig. 8–30a is counterclockwise, the direction of $\boldsymbol{\omega}$ is upward as shown in Fig. 8–30b. If the wheel rotates clockwise, then $\boldsymbol{\omega}$ points in the opposite direction, downward. Note that no part of the rotating body moves in the direction of $\boldsymbol{\omega}$.

Right-hand rule

If the axis of rotation is fixed, then $\boldsymbol{\omega}$ can change only in magnitude. Thus $\boldsymbol{\alpha} = \Delta\boldsymbol{\omega}/\Delta t$ must also point along the axis of rotation. If the rotation is counterclockwise as in Fig. 8–30a, and if the magnitude ω is increasing, then $\boldsymbol{\alpha}$ points upward; but if ω is decreasing (the wheel is slowing down), $\boldsymbol{\alpha}$ points downward. If the rotation is clockwise, $\boldsymbol{\alpha}$ will point downward if ω is increasing, and point upward if ω is decreasing.

Angular momentum, like linear momentum, is a vector quantity. For a symmetrical body (such as a wheel, cylinder, hoop, sphere) rotating about a symmetry axis, we can write the vector angular momentum as

$$\mathbf{L} = I\boldsymbol{\omega}. \qquad \textbf{(8–19)}$$

The angular velocity vector $\boldsymbol{\omega}$ (and therefore also **L**) points along the axis of rotation in the direction given by the right-hand rule (Fig. 8–30b).

The vector nature of angular momentum can be used to explain a number of interesting (and sometimes surprising) phenomena. For example, consider a person standing at rest on a circular platform capable of rotating without friction about an axis through its center (that is, a simplified merry-go-round). If the person now starts to walk along the edge of the platform, Fig. 8–31a, the platform starts rotating in the opposite direction. Why? First of all because the person's foot exerts a force on it; but also because (and this is the most useful analysis here) this is an example of the conservation of angular momentum. If the person starts walking counterclockwise, the person's angular momentum will be pointed upward along the axis of rotation (remember how we defined the direction of $\boldsymbol{\omega}$ using the right-hand rule). The magnitude of the person's angular momentum will be $L = I\omega = (mr^2)(v/r)$, where v is the person's speed (relative to Earth, not the platform), r is his distance from the rotation axis, m is his mass, and his moment of inertia is mr^2 if we consider him a particle (mass concentrated at one point). The platform rotates in the opposite direction, so its angular momentum points downward. If the initial total angular momentum was zero (person and platform at rest), it will remain zero after the person starts walking—that is, the upward angular momentum of the person just balances the oppositely directed downward angular momentum of the platform (Fig. 8–31b), so the total vector angular momentum remains zero. Even though the person exerts a force (and torque) on the platform, and vice versa, these are internal torques (internal to the system consisting of platform plus person). There are no external torques (assuming friction-free bearings of the platform), so the angular momentum remains constant.

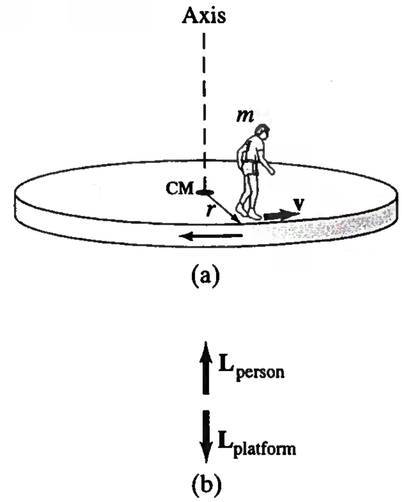
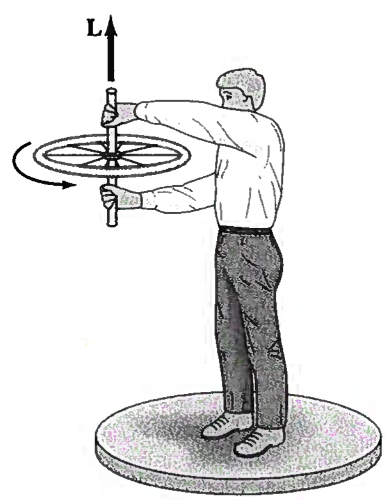


FIGURE 8–31 (a) A person standing on a circular platform, both initially at rest, begins walking along the edge at speed v. The platform, assumed to be mounted on friction-free bearings, begins rotating in the opposite direction, so that the total angular momentum remains zero, as shown in (b).

FIGURE 8–32 Example 8–17.



CONCEPTUAL EXAMPLE 8–17 **Spinning bicycle wheel.** Your physics teacher is holding a spinning bicycle wheel while standing on a stationary frictionless turntable (Fig. 8–32). What will happen if the teacher suddenly flips the bicycle wheel over so that it is spinning in the opposite direction?

RESPONSE The total angular momentum initially is **L** vertically upward, and that's what the system's angular momentum must be afterward since **L** is conserved. Thus, if the wheel's angular momentum afterward is $-\mathbf{L}$ downward, then the angular momentum of teacher plus turntable will have to be $+2\mathbf{L}$ upward. We can safely predict that the teacher will begin spinning around in the same direction the wheel was spinning originally.

SUMMARY

When a rigid body rotates about a fixed axis, each point of the body moves in a circular path. Lines drawn perpendicularly from the rotation axis to various points in the body all sweep out the same angle θ in any given time interval.

Angles are conveniently measured in **radians**, where one radian is the angle subtended by an arc whose length is equal to the radius, or

$$2\pi \text{ rad} = 360°$$

$$1 \text{ rad} \approx 57.3°.$$

Angular velocity, ω, is defined as the rate of change of angular position:

$$\omega = \frac{\Delta\theta}{\Delta t}.$$

All parts of a rigid body rotating about a fixed axis have the same angular velocity at any instant.

Angular acceleration, α, is defined as the rate of change of angular velocity:

$$\alpha = \frac{\Delta\omega}{\Delta t}.$$

The linear velocity v and acceleration a of a point fixed at a distance r from the axis of rotation are related to ω and α by

$$v = r\omega, \qquad a_{\text{tan}} = r\alpha, \qquad a_{\text{R}} = \omega^2 r,$$

where a_{tan} and a_{R} are the tangential and radial (centripetal) components of the linear acceleration, respectively.

The frequency f is related to ω by $\omega = 2\pi f$, and to the period T by $T = 1/f$.

The equations describing uniformly accelerated rotational motion (α = constant) have the same form as for uniformly accelerated linear motion:

$$\omega = \omega_0 + \alpha t; \qquad \theta = \omega_0 t + \tfrac{1}{2}\alpha t^2;$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta; \qquad \overline{\omega} = \frac{\omega + \omega_0}{2}.$$

The dynamics of rotation is analogous to the dynamics of linear motion. Force is replaced by **torque**, τ, which is defined as the product of force times lever arm (perpendicular distance from the line of action of the force to the axis of rotation). Mass is replaced by **moment of inertia**, I, which depends not only on the mass of the body, but also on how the mass is distributed about the axis of rotation. Linear acceleration is replaced by angular acceleration. The rotational equivalent of Newton's second law is then

$$\Sigma\tau = I\alpha.$$

The **rotational kinetic energy** of a body rotating about a fixed axis with angular velocity ω is

$$\text{KE} = \tfrac{1}{2} I\omega^2.$$

For a body both translating and rotating, the total kinetic energy is the sum of the translational KE of the body's CM plus the rotational KE of the body about its CM:

$$\text{KE} = \tfrac{1}{2} M v_{\text{CM}}^2 + \tfrac{1}{2} I_{\text{CM}}\,\omega^2$$

as long as the rotation axis is fixed in direction.

The **angular momentum**, L, of a body about a fixed rotation axis is given by

$$L = I\omega.$$

Newton's second law, in terms of angular momentum, becomes

$$\Sigma\tau = \frac{\Delta L}{\Delta t}.$$

If the net torque on the body is zero, $\Delta L/\Delta t = 0$, so L = constant. This is the **law of conservation of angular momentum** for a rotating body.

QUESTIONS

1. You are standing a known distance from the Statue of Liberty. Describe how you could determine its height using only a meter stick, and without moving from your place.
2. A bicycle odometer (which measures distance traveled) is attached near the wheel hub and is designed for 27-inch wheels. What happens if you use it on a bicycle with 24-inch wheels?
3. Suppose a record turntable rotates at constant angular velocity. Does a point on the rim have radial and/or tangential acceleration? If the turntable's angular velocity increases uniformly, does the point have radial and/or tangential acceleration? For which cases would the magnitude of either of the components of linear acceleration change?
4. If the angular quantities θ, ω, and α were specified in terms of degrees rather than radians, how would Eqs. 8–9 for uniformly accelerated rotational motion have to be altered?
5. Can a small force exert a greater torque than a larger force? Explain.
6. If a force **F** acts on a body such that its lever arm is zero, does it have any effect on the body's motion?
7. Why is it more difficult to do a sit-up with your hands behind your head than when they are outstretched in front of you? A diagram may help you to answer this.
8. Expert bicyclists use very lightweight "sew-up" (tubular) tires. They claim that reducing the mass of the tires is far more significant than an equal reduction in mass elsewhere on the bicycle. Explain why.
9. A 21-speed bicycle has seven sprockets at the rear wheel and three at the pedal cranks. In which gear is it harder to pedal, a small rear sprocket or a large rear sprocket? Why? In which gear is it harder to pedal, a small front sprocket or a large front sprocket? Why?
10. Mammals that depend on being able to run fast have slender lower legs with flesh and muscle concentrated high, close to the body (Fig. 8–33). On the basis of rotational dynamics, explain why this distribution of mass is advantageous.

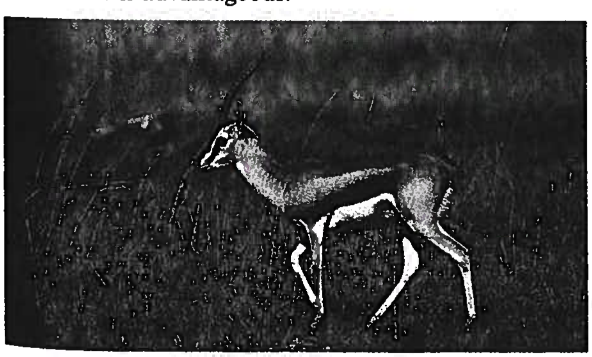


FIGURE 8–33 A gazelle. Question 10.

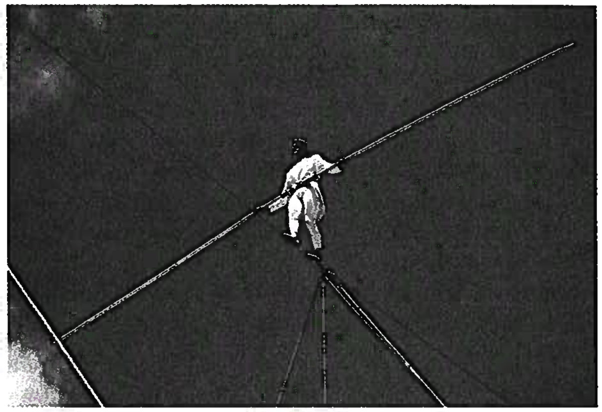


FIGURE 8–34 Question 11.

11. Why do tightrope walkers carry a long, narrow beam (Fig. 8–34)?
12. If the net force on a system is zero, is the net torque also zero? If the net torque on a system is zero, is the net force zero?
13. A stick stands vertically on its end on a frictionless surface. Describe the motion of its CM, and of each end, when it is tipped slightly to one side and falls.
14. Two inclines have the same height but make different angles with the horizontal. The same steel ball is rolled down each incline. On which incline will the speed of the ball at the bottom be greatest? Explain.
15. Two solid spheres simultaneously start rolling (from rest) down an incline. One sphere has twice the radius and twice the mass of the other. Which reaches the bottom of the incline first? Which has the greater speed there? Which has the greater total kinetic energy at the bottom?
16. A sphere and a cylinder have the same radius and the same mass. They start from rest at the top of an incline. Which reaches the bottom first? Which has the greater speed at the bottom? Which has the greater total kinetic energy at the bottom? Which has the greater rotational KE?
17. A cyclist rides over the top of a hill. Is the bicycle's motion rotational, translational, or a combination of both?
18. We claim that momentum and angular momentum are conserved. Yet most moving or rotating bodies eventually slow down and stop. Explain.
19. If there were a great migration of people toward the equator, how would this affect the length of the day?
20. Can the diver of Fig. 8–28 do a somersault without having any initial rotation when she leaves the board?

* **21.** In what direction is the Earth's angular velocity for its daily rotation on its axis?

* **22.** The angular velocity of a wheel rotating on a horizontal axle points west. In what direction is the linear velocity of a point on the top of the wheel? If the angular acceleration points east, describe the tangential linear acceleration of this point. Is the angular speed increasing or decreasing?

* **23.** When a motorcyclist leaves the ground on a jump, if the throttle is left on (so that the rear wheel spins), why does the front of the cycle rise up?

* **24.** Suppose you are standing on the edge of a large rotating turntable. What happens if you walk toward the center?

* **25.** A quarterback leaps into the air to throw a forward pass. As he throws the ball, the upper part of his body rotates. If you look quickly you will notice that his hips and legs rotate in the opposite direction (Fig. 8–35). Explain.

* **26.** Look at the face of a clock with a second hand. In what direction is the angular momentum of the second hand?

FIGURE 8–35 Quarterback in the air, throwing a pass. Question 25.

* **27.** On the basis of the law of conservation of angular momentum, discuss why a helicopter must have more than one rotor (or propeller). Discuss one or more ways the second propeller can operate in order to keep the body stable.

PROBLEMS

SECTION 8–1

1. (I) What are the following angles expressed in radians: (*a*) 30°, (*b*) 57°, (*c*) 90°, (*d*) 360°, and (*e*) 420°? Give as numerical values and as fractions of π.
2. (I) The Sun subtends an angle of about 0.5° to us on the Earth, 150 million km away. What is the radius of the Sun?
3. (I) Eclipses happen on Earth because of an amazing coincidence. Calculate, using the information inside the front cover, the angular diameter (in radians) of the Sun and the angular diameter of the Moon, as seen on Earth.
4. (I) The Eiffel Tower is 300 m tall. When you are standing at a certain place in Paris, it subtends an angle of 6°. How far are you, then, from the Eiffel Tower?
5. (I) A laser beam is directed at the Moon, 380,000 km from Earth. The beam diverges at an angle θ (Fig. 8–36) of 1.8×10^{-5} rad. What diameter spot will it make on the Moon?

(I) A 0.35-m diameter grinding wheel rotates at 1800 rpm. Calculate its angular velocity in rad/s.

(I) What is the linear speed and acceleration of a point on the edge of the grinding wheel in Problem 6?

8. (I) A 33-rpm phonograph record reaches its rated speed 1.8 s after it is turned on. What was the angular acceleration?

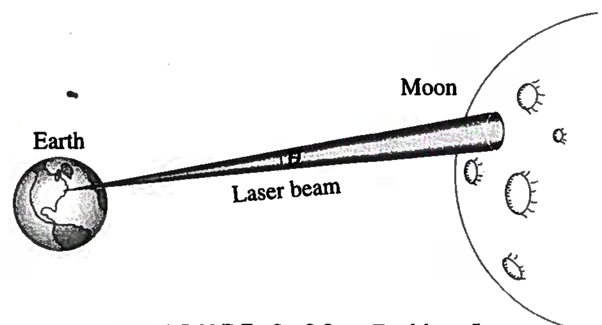


FIGURE 8–36 Problem 5.

9. (I) Calculate the angular velocity of (*a*) the second hand, (*b*) the minute hand, and (*c*) the hour hand, of a clock. State in rad/s. (*d*) What is the angular acceleration in each case?
10. (I) The blades in a blender rotate at a rate of 7500 rpm. When the motor is turned off during operation, the blades slow to rest in 3.0 s. What is the angular acceleration as the blades slow down?
11. (II) A child rolls a ball on a level floor 4.5 m to another child. If the ball makes 15.0 revolutions, what is its diameter?
12. (II) A bicycle with 68-cm-diameter tires travels 7.0 km. How many revolutions do the wheels make?

13. (II) Estimate the angle subtended by the Moon using a ruler and your finger or other object to just blot out the Moon. Describe your measurement and the result obtained and then use it to estimate the diameter of the Moon. The Moon is about 380,000 km from the Earth.

14. (II) Calculate the angular velocity of the Earth (*a*) in its orbit around the Sun, and (*b*) about its axis.

15. (II) What is the linear speed of a point (*a*) on the equator, (*b*) on the Arctic Circle (latitude 66.5° N), and (*c*) at a latitude of 45.0° N, due to the Earth's rotation?

16. (II) How fast (in rpm) must a centrifuge rotate if a particle 7.0 cm from the axis of rotation is to experience an acceleration of 100,000 *g*'s?

17. (II) A 70-cm-diameter wheel accelerates uniformly from 160 rpm to 280 rpm in 4.0 s. Determine (*a*) its angular acceleration, and (*b*) the radial and tangential components of the linear acceleration of a point on the edge of the wheel 2.0 s after it has started accelerating.

18. (II) A record player turntable of radius R_1 is turned by a circular rubber roller of radius R_2 in contact with it at their outer edges. What is the ratio of their angular velocities, ω_1/ω_2?

19. (II) In traveling to the Moon astronauts aboard the Apollo spacecraft put themselves into a slow rotation in order to distribute the Sun's energy evenly. At the start of their trip, they accelerated from no rotation to one revolution every minute during a 10-min time interval. The spacecraft can be thought of as a cylinder with a diameter of 8.5 m. Determine (*a*) the angular acceleration, and (*b*) the radial and tangential components of the linear acceleration of a point on the skin of the ship 5.0 min after it started this acceleration.

SECTIONS 8–2 AND 8–3

20. (I) A phonograph turntable reaches its speed of 33 rpm after making 1.7 revolutions. What was its angular acceleration?

21. (I) A centrifuge accelerates from rest to 15,000 rpm in 220 s. Through how many revolutions did it turn in this time?

22. (I) An automobile engine slows down from 4000 rpm to 1200 rpm in 3.5 s. Calculate (*a*) its angular acceleration, assumed uniform, and (*b*) the total number of revolutions the engine makes in this time.

23. (II) Pilots can be tested for the stresses of flying high-speed jets in a whirling "human centrifuge" which takes 1.0 min to turn through 20 complete revolutions before reaching its final speed. (*a*) What was its angular acceleration (assume constant), and (*b*) what was its final speed in rpm?

24. (II) A 40-cm-diameter wheel accelerates uniformly from 240 rpm to 360 rpm in 6.5 s. How far will a point on the edge of the wheel have traveled in this time?

(II) Starting from the definitions of ω and α, derive Eqs. 8–9 assuming constant angular acceleration.

26. (II) A small rubber wheel is used to drive a large pottery wheel, and they are mounted so that their circular edges touch. If the small wheel has a radius of 2.0 cm and accelerates at the rate of 7.2 rad/s^2, and it is in contact with the pottery wheel (radius 25.0 cm) without slipping, calculate (*a*) the angular acceleration of the pottery wheel, and (*b*) the time it takes the pottery wheel to reach its required speed of 65 rpm.

27. (II) The tires of a car make 65 revolutions as the car reduces its speed uniformly from 100 km/h to 50 km/h. The tires have a diameter of 0.80 m. (*a*) What was the angular acceleration? (*b*) If the car continues to decelerate at this rate, how much more time is required for it to stop?

28. (III) A wheel, starting from rest, undergoes uniform angular acceleration α about its fixed axle. (*a*) Write the components of the linear acceleration, a_{tan} and a_R, for a point P which is a distance r from the axle in terms of α, r, and time t. (*b*) Let ϕ be the angle between the linear acceleration vector, **a**, and the line drawn between P and the axis. Express ϕ in terms of the total number of revolutions of the wheel, N.

SECTION 8–4

29. (I) What is the maximum torque exerted by a 55-kg person riding a bike if the rider puts all her weight on each pedal when climbing a hill? The pedals rotate in a circle of radius 17 cm.

30. (I) A person exerts a force of 45 N on the end of a door 84 cm wide. What is the magnitude of the torque if the force is exerted (*a*) perpendicular to the door, and (*b*) at a 60° angle to the face of the door?

31. (II) Calculate the net torque about the axle of the wheel shown in Fig. 8–37. Assume that a friction torque of 0.40 m·N opposes the motion.

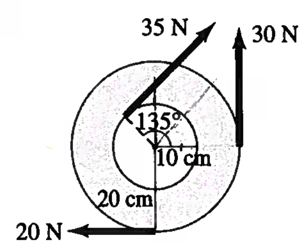


FIGURE 8–37 Problem 31.

32. (II) If the coefficient of static friction between tires and pavement is 0.75, calculate the minimum torque that must be applied to the 66-cm-diameter tire of a 1080-kg automobile in order to "lay rubber" (make the wheels spin, slipping as the car accelerates). Assume each wheel supports an equal share of the weight.

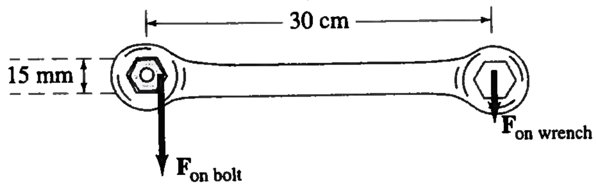


FIGURE 8–38 Problem 33.

33. (II) The bolts on the cylinder head of an engine require tightening to a torque of 80 m·N. If a wrench is 30 cm long, what force perpendicular to the wrench must the mechanic exert at its end? If the six-sided bolt head is 15 mm in diameter, estimate the force applied near each of the six points by a socket wrench (Fig. 8–38).

SECTIONS 8–5 AND 8–6

34. (I) Calculate the moment of inertia of a 12.2-kg sphere of radius 0.623 m when the axis of rotation is through its center.
35. (I) Calculate the moment of inertia of a 66.7-cm-diameter bicycle wheel. The rim and tire have a combined mass of 1.25 kg. The mass of the hub can be ignored (why?).
36. (II) Calculate the moment of inertia of the array of point objects shown in Fig. 8–39 about (*a*) the vertical axis, and (*b*) the horizontal axis. Assume the objects are wired together by very light rigid pieces of wire. About which axis would it be harder to accelerate this array? In Fig. 8–39, $m = 1.8$ kg and $M = 3.1$ kg. The array is rectangular and it is split through the middle by the horizontal axis.

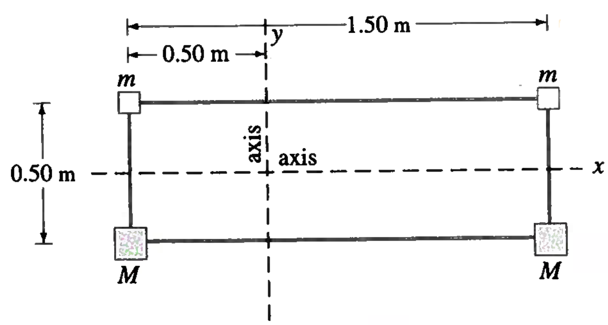


FIGURE 8–39 Problem 36.

37. (II) An oxygen molecule consists of two oxygen atoms whose total mass is 5.3×10^{-26} kg and whose moment of inertia about an axis perpendicular to the line joining the two atoms, midway between them, is 1.9×10^{-46} kg·m². Estimate, from these data, the effective distance between the atoms.
38. (II) A small 1.05-kg ball on the end of a light rod is rotated in a horizontal circle of radius 0.900 m. Calculate (*a*) the moment of inertia of the system about the axis of rotation, and (*b*) the torque needed to keep the ball rotating at constant angular velocity if air resistance exerts a force of 0.0800 N on the ball.

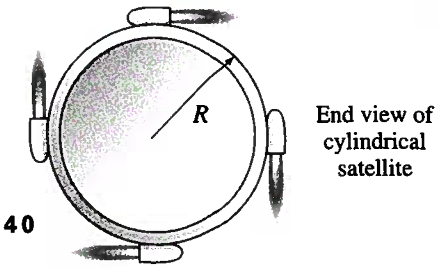


FIGURE 8–40 Problem 39.

39. (II) In order to get a flat uniform cylindrical satellite spinning at the correct rate, engineers fire four tangential rockets as shown in Fig. 8–40. If the satellite has a mass of 2600 kg and a radius of 3.0 m, what is the required steady force of each rocket if the satellite is to reach 30 rpm in 5.0 min?
40. (II) A grinding wheel is a uniform cylinder with a radius of 8.50 cm and a mass of 0.580 kg. Calculate (*a*) its moment of inertia about its center, and (*b*) the applied torque needed to accelerate it from rest to 1500 rpm in 5.00 s if it is known to slow down from 1500 rpm to rest in 55.0 s.
41. (II) A softball player swings a bat, accelerating it from rest to 3.0 rev/s in a time of 0.20 s. Approximate the bat as a 2.2-kg uniform rod of length 0.95 m, and compute the torque the player applies to one end of it.
42. (II) A day-care worker pushes tangentially on a small hand-driven merry-go-round and is able to accelerate it from rest to a spinning rate of 20 rpm in 10.0 s. Assume the merry-go-round is a disk of radius 2.5 m and has a mass of 800 kg, and two children (each with a mass of 25 kg) sit opposite each other on the edge. Calculate the torque required to produce the acceleration, neglecting frictional torque. What force is required?
43. (II) A centrifuge rotor rotating at 10,000 rpm is shut off and is eventually brought to rest by a frictional torque of 1.20 m·N. If the mass of the rotor is 4.80 kg and it can be approximated as a solid cylinder of radius 0.0710 m, through how many revolutions will the rotor turn before coming to rest, and how long will it take?
44. (II) The forearm in Fig. 8–41 accelerates a 3.6-kg ball at 7.0 m/s² by means of the triceps muscle, as shown. Calculate (*a*) the torque needed, and (*b*) the force that must be exerted by the triceps muscle. Ignore the mass of the arm.

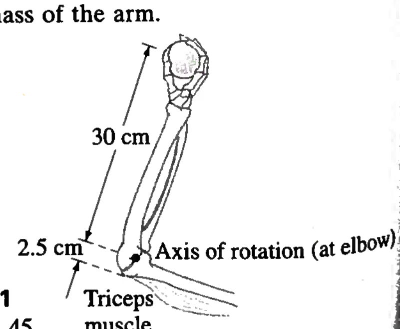


FIGURE 8–41 Problems 44 and 45.

45. (II) Assume that a 1.50-kg ball is thrown solely by the action of the forearm, which rotates about the elbow joint under the action of the triceps muscle, Fig. 8–41. The ball is accelerated from rest to 10.0 m/s in 0.350 s, at which point it is released. Calculate (*a*) the angular acceleration of the arm, and (*b*) the force required of the triceps muscle. Assume that the forearm has a mass of 3.70 kg and rotates like a uniform rod about an axis at its end.

46. (II) A helicopter rotor blade can be considered a long thin rod, as shown in Fig. 8–42. If each of the three rotor helicopter blades is 3.75 m long and has a mass of 160 kg, calculate the moment of inertia of the three rotor blades about the axis of rotation. How much torque must the motor apply to bring the blades up to a speed of 5.0 rev/s in 8.0 s?

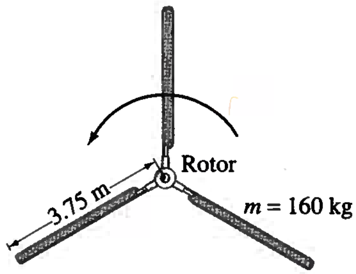


FIGURE 8–42 Problem 46.

47. (II) The radius of the roll of paper shown in Fig. 8–43 is 7.6 cm and its moment of inertia is $I = 2.9 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$. A force of 3.2 N is exerted on the end of the roll for 1.3 s, but the paper does not tear so it begins to unroll. A constant friction torque of 0.11 m·N is exerted on the roll which gradually brings it to a stop. Assuming that the paper's thickness is negligible, calculate (*a*) the length of paper that unrolls during the time that the force is applied (1.3 s) and (*b*) the length of paper that unrolls from the time the force ends to the time where the roll has stopped moving.

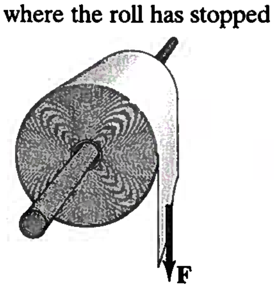


FIGURE 8–43 Problem 47.

48. (III) An *Atwood's machine* consists of two masses, m_1 and m_2, which are connected by a massless inelastic cord that passes over a pulley, Fig. 8–44. If the pulley has radius R and moment of inertia I about its axle, determine the acceleration of the masses m_1 and m_2, and compare to the situation in which the moment of inertia of the pulley is ignored. [*Hint:* The tensions F_{T1} and F_{T2} are not necessarily equal.]

49. (III) A hammer thrower accelerates the hammer (mass = 7.30 kg) from rest within four full turns (revolutions) and releases it at a speed of 28.0 m/s. Assuming a uniform rate of increase in angular velocity and a radius of 1.20 m, calculate (*a*) the angular acceleration, (*b*) the (linear) tangential acceleration, (*c*) the centripetal acceleration just before release, (*d*) the net force being exerted on the hammer by the athlete just before release, and (*e*) the angle of this force with respect to the radius of the circular motion.

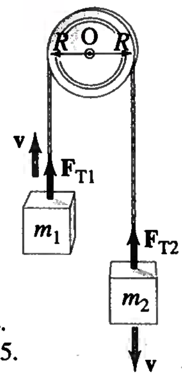


FIGURE 8–44 Atwood's machine. Problems 48 and 55.

SECTION 8–7

50. (I) A bowling ball of mass 7.3 kg and radius 9.0 cm rolls without slipping down a lane at 4.3 m/s. Calculate its total kinetic energy.

51. (I) A centrifuge rotor has a moment of inertia of $3.15 \times 10^{-2}\ \text{kg}\cdot\text{m}^2$. How much energy is required to bring it from rest to 8000 rpm?

52. (II) Estimate the kinetic energy of the Earth with respect to the Sun as the sum of two terms, (*a*) that due to its daily rotation about its axis, and (*b*) that due to its yearly revolution about the Sun. [Assume the Earth is a uniform sphere, mass = 6.0×10^{24} kg, radius = 6.4×10^6 m, and is 1.5×10^8 km from the Sun.]

53. (II) A merry-go-round has a mass of 1640 kg and a radius of 8.20 m. How much net work is required to accelerate it from rest to a rotation rate of one revolution in 8.00 s? (Assume it is a solid cylinder.)

54. (II) (*a*) Calculate the translational and rotational speeds of a sphere (radius 20.0 cm and mass 1.20 kg), that rolls without slipping down a 30.0° incline that is 10.0 m long, when it reaches the bottom. Assume it started from rest. (*b*) What is its ratio of translational to rotational KE at the bottom? Try to avoid putting in numbers until the end so you can answer: (*c*) do your answers in (*a*) and (*b*) depend on the radius of the sphere or its mass?

55. (III) Two masses, $m_1 = 18.0$ kg and $m_2 = 26.5$ kg, are connected by a rope that hangs over a pulley (as in Fig. 8–44). The pulley is a uniform cylinder of radius 0.260 m and mass 7.50 kg. Initially, m_1 is on the ground and m_2 rests 3.00 m above the ground. If the system is now released, use conservation of energy to determine the speed of m_2 just before it strikes the ground. Assume the pulley is frictionless.

56. (III) A 3.30-m-long pole is balanced vertically on its tip. It is given a tiny push. What will be the speed of the upper end of the pole just before it hits the ground? Assume the lower end does not slip.

SECTION 8–8

57. (I) What is the angular momentum of a 0.210-kg ball rotating on the end of a string in a circle of radius 1.10 m at an angular speed of 10.4 rad/s?

58. (I) A person stands, hands at the side, on a platform that is rotating at a rate of 1.30 rev/s. If the person now raises his arms to a horizontal position, Fig. 8–45, the speed of rotation decreases to 0.80 rev/s. (*a*) Why does this occur? (*b*) By what factor has the moment of inertia of the person changed?

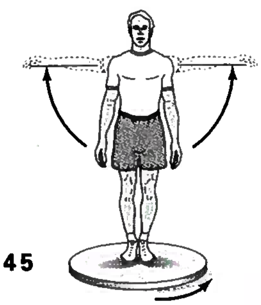


FIGURE 8–45
Problem 58.

59. (I) A diver (such as the one shown in Fig. 8–28) can reduce her moment of inertia by a factor of about 3.5 when changing from the straight position to the tuck position. If she makes two rotations in 1.5 s when in the tuck position, what is her angular speed (rev/s) when in the straight position?

60. (I) A figure skater during her finale can increase her rotation rate from an initial rate of 1.0 rev every 2.0 s to a final rate of 3.0 rev/s. If her initial moment of inertia was 4.6 kg·m^2, what is her final moment of inertia? How does she physically accomplish this change?

61. (II) Hurricanes can involve winds in excess of 120 km/h. Make a crude estimate of (*a*) the energy, and (*b*) the angular momentum, of such a hurricane, approximating it as a rigidly rotating uniform cylinder of air (density 1.3 kg/m^3) of radius 100 km and height 4.0 km.

62. (II) (*a*) What is the angular momentum of a figure skater spinning (with arms in close to her body) at 3.5 rev/s, assuming her to be a uniform cylinder with a height of 1.5 m, a radius of 15 cm, and a mass of 55 kg. (*b*) How much torque is required to slow her to a stop in 5.0 s, assuming she does *not* move her arms?

63. (II) Determine the angular momentum of the Earth (*a*) about its rotation axis (assume the Earth is a uniform sphere), and (*b*) in its orbit around the Sun (treat the Earth as a particle orbiting the Sun). The Earth has mass = 6.0×10^{24} kg, radius = 6.4×10^6 m, and is 1.5×10^8 km from the Sun.

64. (II) A nonrotating cylindrical disk of moment of inertia I is dropped onto an identical disk rotating at angular speed ω. Assuming no external torques, what is the final common angular speed of the two disks?

65. (II) A uniform disk, such as a record turntable, turns at 7.0 rev/s around a frictionless spindle. A nonrotating rod, of the same mass as the disk and length equal to the disk's diameter, is dropped onto the freely spinning disk. They then both turn around the spindle with their centers superposed, Fig. 8–46. What is the angular velocity in rev/s of the combination?

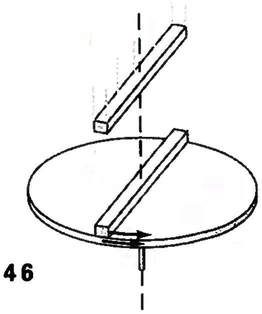


FIGURE 8–46
Problem 65.

66. (II) A person of mass 75 kg stands at the center of a rotating merry-go-round platform of radius 3.0 m and moment of inertia 1000 kg·m^2. The platform rotates without friction with angular velocity 2.0 rad/s. The person walks radially to the edge of the platform. (*a*) Calculate the angular velocity when the person reaches the edge. (*b*) Compare the rotational kinetic energies of the system of platform plus person before and after the person's walk.

67. (II) An asteroid of mass 1.0×10^5 kg, traveling at a speed of 30 km/s relative to the Earth, hits the Earth at the equator. It hits the Earth tangentially and in the direction of Earth's rotation. Use angular momentum to estimate the fractional change in the angular speed of the Earth as a result of the collision.

68. (II) A 4.2-m-diameter merry-go-round is rotating freely with an angular velocity of 0.80 rad/s. Its total moment of inertia is 1760 kg·m^2. Four people standing on the ground, each of mass 65 kg, suddenly step onto the edge of the merry-go-round. What is the angular velocity of the merry-go-round now? What if the people were on it initially and then jumped off in a radial direction (relative to the merry-go-round)?

69. (II) Suppose our Sun eventually collapses into a white dwarf, in the process losing about half its mass and winding up with a radius 1.0 percent of its existing radius. What would its new rotation rate be? (Take the Sun's current period to be about 30 days.) What would be its final KE in terms of its initial KE of today?

*SECTION 8–9

*70. (II) Suppose a 55-kg person stands at the edge of a 6.5-m-diameter merry-go-round turntable that is mounted on frictionless bearings and has a moment of inertia of 1700 kg·m^2. The turntable is at rest initially, but when the person begins running at a speed of 3.8 m/s (with respect to the turntable) around its edge, the turntable begins to rotate in the opposite direction. Calculate the angular velocity of the turntable.

*71. (II) A person stands on a platform, initially at rest, that can rotate freely without friction. The moment of inertia of the person plus the platform is I_P. The person holds a spinning bicycle wheel with axis horizontal. The wheel has moment of inertia I_W and angular velocity ω_W. What will be the angular velocity ω_P of the platform if the person moves the axis of the wheel so that it points (*a*) vertically upward, (*b*) at a 60° angle to the vertical, (*c*) vertically downward? (*d*) What will ω_P be if the person reaches up and stops the wheel in part (*a*)?

GENERAL PROBLEMS

72. A large spool of rope stands on the ground with the end of the rope lying on the top edge of the spool. A person grabs the end of the rope and walks a distance L, holding onto it, Fig. 8–47. The spool rolls behind the person without slipping. What length of rope unwinds from the spool? How far does the spool's CM move?

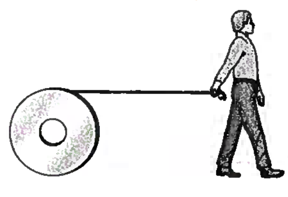


FIGURE 8–47 Problem 72.

73. The Moon orbits the Earth so that the same side always faces the Earth. Determine the ratio of its spin angular momentum (about its own axis) to its orbital angular momentum. (In the latter case, treat the Moon as a particle orbiting the Earth.)

74. A cyclist accelerates from rest at a rate of $1.00\,\mathrm{m/s^2}$. How fast will a point on the rim of the tire (diameter = 68 cm) at the top be moving after 3.0 s? [*Hint:* At any moment, the lowest point on the tire is in contact with the ground and is at rest—see Fig. 8–48.]

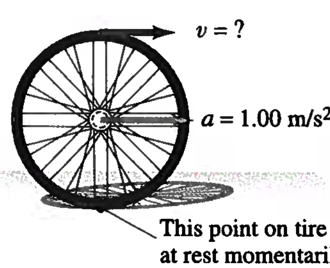


FIGURE 8–48 Problem 74.

75. (*a*) A yo-yo is made of two solid cylindrical disks, each of mass 0.050 kg and diameter 0.075 m, joined by a (concentric) thin solid cylindrical hub of mass 0.0050 kg and diameter 0.010 m. Use conservation of energy to calculate the linear speed of the yo-yo when it reaches the end of its 1.0-m-long string, if it is released from rest. (*b*) What fraction of its kinetic energy is rotational?

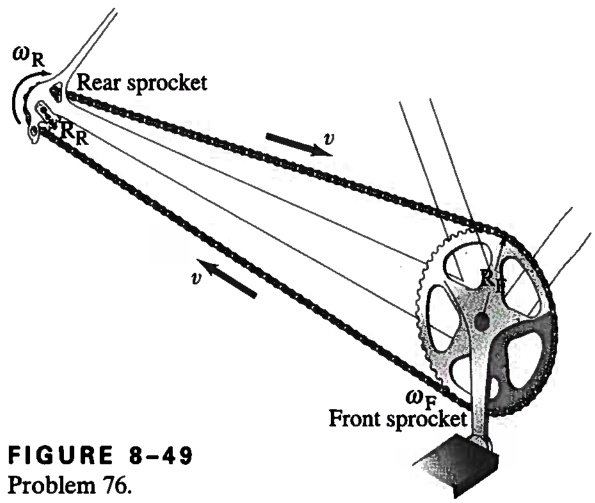


FIGURE 8–49 Problem 76.

76. (a) How is the angular speed of the rear wheel (ω_R) of a bicycle related to that of the pedals and front sprocket (ω_F)? That is, derive a formula for ω_R/ω_F. Let N_F and N_R be the number of teeth on the front and rear sprockets, respectively. The teeth are spaced equally on all sprockets so that the chain meshes properly. See Fig. 8–49. Then evaluate the ratio ω_R/ω_F when (*b*) the front and rear sprockets have 52 and 13 teeth, respectively, and (*c*) when they have 42 and 28.

77. Suppose a star the size of our Sun, but of mass 8.0 times as great, were rotating at a speed of 1.0 revolution every 10 days. If it were to undergo gravitational collapse to a neutron star of radius 10 km, losing $\frac{3}{4}$ of its mass in the process, what would its rotation speed be? Assume the star is a uniform sphere at all times and loses no angular momentum in the process.

78. One possibility for a low-pollution automobile is for it to use energy stored in a heavy rotating flywheel. Suppose such a car has a total mass of 1400 kg, uses a 1.50-m diameter uniform cylindrical flywheel of mass 240 kg, and should be able to travel 300 km without needing a flywheel "spinup." (*a*) Make reasonable assumptions (average frictional retarding force = 500 N, twenty acceleration periods from rest to 90 km/h, equal uphill and downhill—assuming during downhill, energy can be put back into the flywheel), and show that the total energy needed to be stored in the flywheel is about 1.6×10^8 J. (*b*) What is the angular velocity of the flywheel when it has a full "energy charge"? (*c*) About how long would it take a 150-hp motor to give the flywheel a full energy charge before a trip?

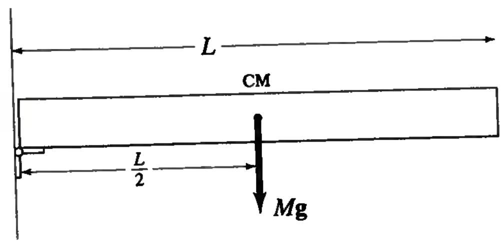


FIGURE 8–50 Problem 80.

79. A hollow cylinder (hoop) is rolling on a horizontal surface at speed $v = 4.3\ \mathrm{m/s}$ when it reaches a 15° incline. (*a*) How far along the incline will it go? (*b*) How long will it be on the incline before it arrives back at the bottom?

80. A uniform rod of mass M and length L can pivot freely (i.e., we ignore friction) about a hinge attached to a wall, as in Fig. 8–50. The rod is held horizontally and then released. At the moment of release, determine (*a*) the angular acceleration of the rod and (*b*) the linear acceleration of the tip of the rod. Assume that the force of gravity acts at the center of mass of the rod, as shown. [*Hint*: See Fig. 8–20g.]

81. A wheel of mass M has radius R. It is standing vertically on the floor, and we want to exert a horizontal force F at its axle so that it will climb a step against which it rests (Fig. 8–51). The step has height h, where $h < R$. What minimum force F is needed?

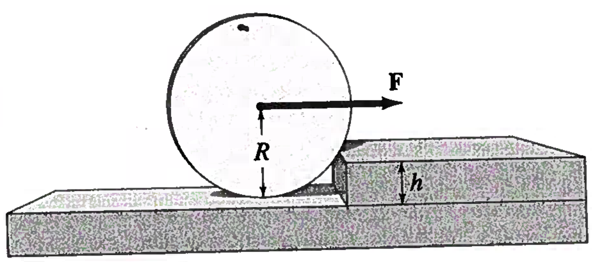


FIGURE 8–51 Problem 81.

82. A bicyclist traveling with speed $v = 4.2\ \mathrm{m/s}$ on a flat road is making a turn with a radius $r = 6.4\ \mathrm{m}$. The forces acting on the cyclist and cycle are the normal force ($\mathbf{F}_N$) and friction force ($\mathbf{F}_{fr}$) exerted by the road on the tires, and $m\mathbf{g}$, the total weight of the cyclist and cycle (see Fig. 8–52). (*a*) Explain carefully why the angle θ the bicycle makes with the vertical (Fig. 8–52) must be given by $\tan\theta = F_{fr}/F_N$ if the cyclist is to maintain balance. (*b*) Calculate θ for the values given. [*Hint*: Consider the "circular" translational motion of the bicycle and rider.] (*c*) If the coefficient of static friction between tires and road is $\mu_s = 0.70$, what is the minimum turning radius?

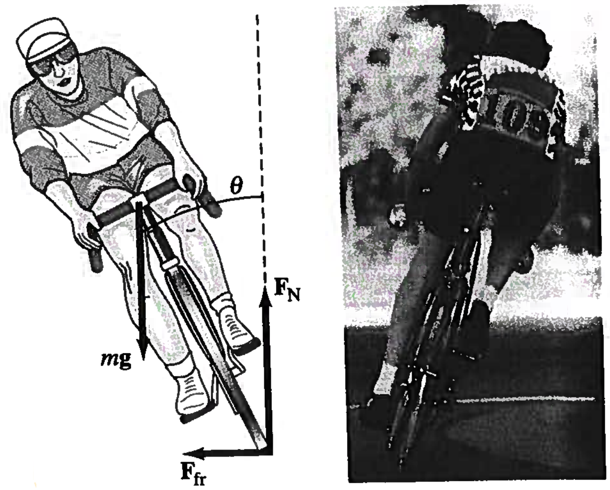


FIGURE 8–52 Problem 82.

83. A marble of mass m and radius r rolls along the looped rough track of Fig. 8–53. What is the minimum value of h if the marble is to reach the highest point of the loop without leaving the track? Assume $r \ll R$, and ignore frictional losses.

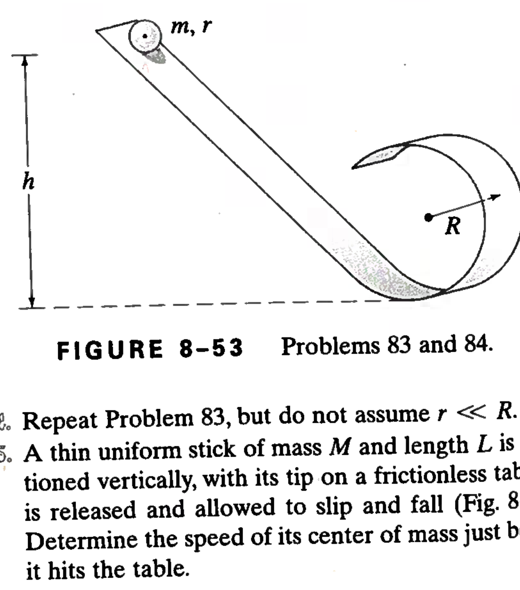


FIGURE 8–53 Problems 83 and 84.

84. Repeat Problem 83, but do not assume $r \ll R$.

85. A thin uniform stick of mass M and length L is positioned vertically, with its tip on a frictionless table. It is released and allowed to slip and fall (Fig. 8–54). Determine the speed of its center of mass just before it hits the table.

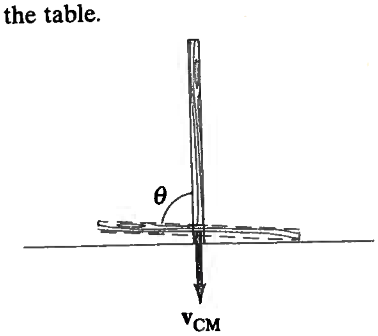


FIGURE 8–54 Problem 85.